Vol. 81, No. 5, December 2008

MATHEMATICS MAGAZINE

Editorial License

- Tiling with Right Trominoes
- The Mystery of Robert Adrain
- Touching the $\mathbb{Z}_2$ in Three-Dimensional Rotations

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The **cover image** displays my license plate as I figuratively drive off into the sunset. Thanks to Don DeLand for suggesting the apt caption. This represents my farewell as editor of the MAGAZINE. The experience has been exciting, exhilarating, and exhausting. I have found myself overjoyed and overwhelmed. I am happy that I chose to take on this task, and I am happy that my term is coming to a close. A special note of appreciation goes to Frank Farris as he returns to the editorial office for Volume 82 beginning with the February 2009 issue.

AUTHORS

Carlos Ueno received his BS from the Universidad Complutense de Madrid and his MS from the University of Toronto. He is currently a teacher of Mathematics at IES Jandía (Fuerteventura, Canary Islands, Spain), where he tries to transmit the beauty of math to his teenage students. He is also planning to complete a PhD program in mathematics, working on problems related to the study of images of real polynomial maps in Euclidean spaces. In his spare time Carlos enjoys the good company of his school colleagues, the wonderful beaches of Fuerteventura, and the exploration of virtual worlds through the Internet.

Frank Swetz is Professor Emeritus of Mathematics and Education at the Pennsylvania State University. At present, he is co-editor, with Victor Katz, of the MAA e-journal *Convergence*. His concerns with humanistic mathematics teaching have led him into studies about the history of mathematics and ethnomathematics. His most recent book is *The Legacy of the Luoshu: a 4000 Year Search for the Meaning of the Magic Square of Order Three*, A. K. Peters (2008). In his spare time he grows vegetables and trains Mongolian fighting crickets.

Vesna Stojanoska is a graduate student at Northwestern University, studying algebraic topology; more precisely, she is trying to understand aspects of stable homotopy theory. She began untangling braids and otherwise working on this article while she was an undergraduate at the American University in Bulgaria.

Orlin Stoytchev is a professor at the American University in Bulgaria. He received his PhD from the Mathematics Department of Virginia Tech. His interests can be summarized as "the different aspects of symmetries in mathematics and physics." He has published works on Von Neumann algebras and on representations of finite- and infinite-dimensional Lie groups and their algebras. The idea for the present article came from an attempt to find an accessible way to demonstrate a rather nontrivial topological fact to his students.

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ARTICLES

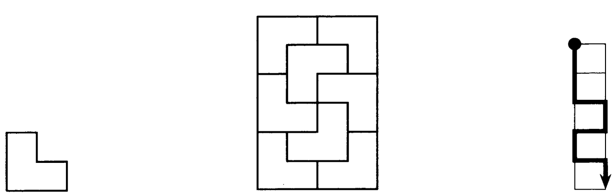
Matrices and Tilings with Right Trominoes

CARLOS UENO
IES Cruz de Piedra
35014 Las Palmas de Gran Canaria, SPAIN
carlos.ueno@terra.es

In his book [**1**], Solomon W. Golomb states that the problem of determining how many ways a $4 \times n$ rectangle can be tiled by right trominoes "appears to be a challenging problem with reasonable hope of an attainable solution." This problem was solved in [**5**] by using generating functions, and similar results have been obtained in [**6**] by S. Heubach, P. Chinn, and P. Callahan, who considered the problem of tiling rectangles with right and straight trominoes. Other beautiful results on right trominoes can be found in [**2**], [**3**], or [**4**]. In this article we present a matrix approach to the suggestion made by Golomb, which has the advantage of being applicable to arbitrary $m \times n$ rectangles—although it also has the drawback of using rather large matrices. The main result (see Theorem 1), though not especially difficult, seems to have passed unnoticed so far. With the help of mathematical software packages such as *Mathematica* we are able to find, for example, the number of different tilings for squares of side a multiple of 3 up to the 12×12 case, as well as generating functions for the number of tilings of rectangles with right trominoes. We also state some results about the tilability of a family of regions called strips, which include rectangles as a particular case.

Basic notation and definitions

A *right tromino* is a shape made up of three 1×1 squares as shown in FIGURE 1a. A *tiling* of an $m \times n$ rectangle by right trominoes consists of a complete covering of the rectangle with these trominoes so that there are no overlappings among them and each tromino is placed "nicely" on it—if we consider the rectangle as formed by mn squares of unit side, each tromino is placed covering perfectly three of these squares. From now on, whenever we use the word *tiling* we will mean a tiling with right trominoes. Our objective in this article is to compute the number of different tilings that rectangles and related regions, which we call *strips*, can have.



(a) Right tromino (b) Tiling (c) 5-binary path

Figure 1 Basic concepts.

Let us start by defining an *m-binary path* as a descending path from a top corner to a bottom corner of an $m \times 1$ rectangle, in such a way that it goes along the sides of the unit squares forming such a rectangle. We always assume that the first and last segments are vertical (see FIGURE 1c). We can see that the vertical edges of these paths can be on the left side or the right side of the hosting rectangle. If we assign the value 0 to its left side and the value 1 to its right side, each m-binary path can be associated to an m-bit string, as shown in FIGURE 2.

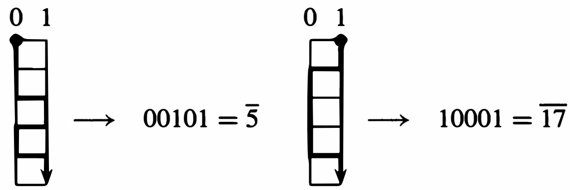


Figure 2 Turning m-binary paths into m-bit strings.

Conversely, each m-bit string corresponds to a unique m-binary path. Therefore, there are 2^m different m-binary paths, and each one can be identified with a number between 0 and $2^m - 1$, written in its binary form. In the rest of this article, we identify an m-binary path and its associated m-bit string with the symbol $\bar{j}$, $0 \leq j \leq 2^m - 1$, where the binary representation of j gives the corresponding path—whenever we drop the bar in $\bar{j}$ we will be referring to the number instead of the path it represents.

Now, let us consider two consecutive m-binary paths $\bar{i}$ and $\bar{j}$ in an $m \times 2$ rectangle, which can be considered as formed by two $m \times 1$ consecutive rectangles (see FIGURE 3).

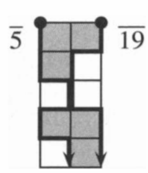


Figure 3 Two consecutive 5-binary paths.

These paths bound a region inside the rectangle, which can be covered or not with right trominoes. We write $\langle \bar{i}, \bar{j} \rangle_m$ to denote the number of different tilings by right trominoes of the region bounded by $\bar{i}$ and $\bar{j}$. As an example, FIGURE 3 illustrates the fact that $\langle \bar{5}, \overline{19} \rangle_5 = 1$, since there is exactly one way to tile the region consisting of the 6 grey cells.

Given a nonnegative integer m, we define the *transfer matrix* G_m as the matrix whose coefficients are $G_m[i, j] = \langle \bar{i}, \bar{j} \rangle_m$—we refer to the coefficient at position $[i, j]$ of any matrix A as $A[i, j]$. A special case arises whenever $\bar{i} = \overline{2^m - 1}$ and $\bar{j} = \bar{0}$, because these consecutive m-binary paths do coincide, and the region bounded by them does not contain any cell. In this situation we set $\langle \overline{2^m - 1}, \bar{0} \rangle_m = 1$, meaning by this that there is just one way to tile this null-area region, which consists in not placing any right tromino on it. Also, for the degenerate case $m = 0$ a similar argument leads us to set $G_0 = [1]$ as most convenient. The three next matrices in this sequence are

$$G_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We invite the reader to check the validity of some coefficients in these matrices, in order to get more familiar with their meaning. The matrices $\{G_m\}$ are our object of study in the next section, for they constitute our main tool to compute the number of different tilings for rectangles and other related regions.

Transfer matrices

We now consider m-binary paths in an $m \times n$ rectangle $\mathbf{R}$. If we consider such a rectangle as the union of n consecutive $m \times 1$ rectangles (see FIGURE 4), then $\overline{w}_k$ represents an m-binary path in the kth unit-width rectangle.

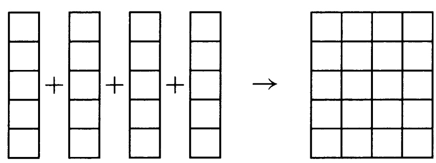


Figure 4 $m \times 1$ rectangles forming an $m \times n$ rectangle.

Given an m-binary path $\overline{w}_k$ and a tiling T in $\mathbf{R}$, we say that $\overline{w}_k$ and T are *compatible* if the path goes along the edges of the trominoes forming the tiling T. For example, the tiling shown in FIGURE 1b is compatible with the path $\overline{0}$ in the first column, with the path $\overline{37}$ in the second column, with the path $\overline{22}$ in the third column and with the path $\overline{63}$ in the fourth column. It is not hard to observe the following fact.

LEMMA 1. *Let* $\mathbf{R}$ *be an* $m \times n$ *rectangle. Then, each tiling with right trominoes determines a unique set of* n *consecutive* m*-binary paths* $\overline{w}_1, \ldots, \overline{w}_n$, *all of them compatible with the tiling.*

Proof. Let us consider the vertical sides $l_0, l_1, \ldots, l_n$ of the n consecutive $m \times 1$ rectangles that $\mathbf{R}$ has. We can partition this rectangle in regions $\mathbf{R}_k$, $1 \le k \le n-1$, each one formed by the union of the trominoes with interiors intersecting l_k (see FIGURE 5).

The interior of $\mathbf{R}_k$ coincides with the interior of the region bounded by two compatible and consecutive m-binary paths $\overline{w}_k$ and $\overline{w}_{k+1}$. Moreover, when considering the next region $\mathbf{R}_{k+1}$ and its bounding paths $\overline{w}'_{k+1}$ and $\overline{w}'_{k+2}$ we must have $\overline{w}_{k+1} = \overline{w}'_{k+1}$, for obvious reasons. Therefore, we can associate to the tiling T an ordered set of compatible and consecutive m-binary paths $\{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n\}$, where we have $\overline{w}_1 = \overline{0}$, $\overline{w}_n = \overline{2^m - 1}$. To see uniqueness, let us suppose there are different sets $\{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n\}$ and $\{\overline{w}'_1, \overline{w}'_2, \ldots, \overline{w}'_n\}$ of consecutive m-binary paths for the same tiling T. Then we would have for some k that $\overline{w}_k \neq \overline{w}'_k$, but this is impossible for that would imply that there is at least one complete tromino in the region bounded by these different paths, both included in the same $m \times 1$ rectangle, and this cannot happen. ∎

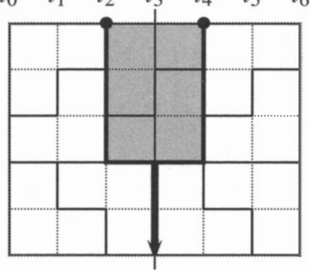


Figure 5 The region $\mathbf{R}_3$ is bounded by two m-binary paths compatible with the tiling.

In order to take full advantage of the sequence of matrices $\{G_m\}$, we need now to generalize our study to a wider family of regions, which include rectangles as a special case.

DEFINITION 1. *An m-strip of order $n \geq 2$ bounded by m-binary paths $\overline{w}_1$ and $\overline{w}_n$ is the region in an $m \times n$ rectangle bounded on the left by $\overline{w}_1$ and on the right by $\overline{w}_n$. We refer to it as $S_m(\overline{w}_1, \overline{w}_n, n)$, and we write $N(S_m(\overline{w}_1, \overline{w}_n, n))$ to denote its number of different tilings.*

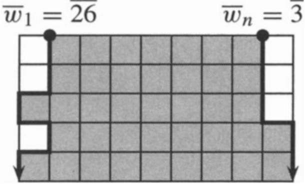


Figure 6 The strip $S_5(\overline{26}, \overline{3}, 9)$.

In particular, an $m \times n$ rectangle can be considered either as the strip $S_m(\overline{0}, \overline{2^m - 1}, n)$, or as the strip $S_m(\overline{0}, \overline{0}, n + 1)$. Clearly Lemma 1 can be generalized to tilings of m-strips.

The next result is straightforward.

LEMMA 2. *Given any m-binary path $\overline{w}_{n-1}$ in the $(n-1)$th column of an $m \times n$ rectangle $\mathbf{R}$, with $n \geq 3$, the number of tilings of the strip $S_m(\overline{w}_1, \overline{w}_n, n)$ that are compatible with $\overline{w}_{n-1}$ is given by the product*

$$N(S_m(\overline{w}_1, \overline{w}_{n-1}, n-1)) \cdot \langle \overline{w}_{n-1}, \overline{w}_n \rangle_m.$$

Notice that when $\overline{w}_{n-1}$ runs through all the possible m-binary paths, the sum of these products is precisely $N(S_m(\overline{w}_1, \overline{w}_n, n))$. Now we are ready to give a full meaning to our family $\{G_m\}$ of transfer matrices.

PROPOSITION 1. *The number of different tilings of the strip $S_m(\overline{w}_1, \overline{w}_n, n)$, $n \geq 2$, is given by the coefficient $G_m^{n-1}[w_1, w_n]$.*

Proof. We use induction on the order n of the strip. For $n = 2$, let us consider two consecutive m-binary paths $\overline{w}_1$ and $\overline{w}_2$ in a $m \times 2$ rectangle. Then, from the definition of the matrix G_m we have

$$G_m[w_1, w_2] = \langle \overline{w}_1, \overline{w}_2 \rangle_m = N(S_m(\overline{w}_1, \overline{w}_2, 2)).$$

Now let us assume that the result is valid for a natural number $n-2 \geq 2$. Then

$$\begin{aligned} G_m^{n-1}[w_1, w_n] &= \sum_{0 \leq w_{n-1} \leq 2^m-1} G_m^{n-2}[w_1, w_{n-1}] \cdot G_m[w_{n-1}, w_n] \\ &= \sum_{0 \leq w_{n-1} \leq 2^m-1} N(S_m(\overline{w}_1, \overline{w}_{n-1}, n-1)) \cdot \langle \overline{w}_{n-1}, \overline{w}_n \rangle_m \\ &= N(S_m(\overline{w}_1, \overline{w}_n, n)), \end{aligned}$$

and therefore the statement is also true for n. ■

With respect to rectangles, the number of different tilings in an $m \times n$ rectangle can be expressed either as $G_m^{n-1}[0, 2^m-1]$ or as $G_m^n[0, 0]$. We will use both possibilities indiscriminately.

Main result

In this section our goal is to give an explicit expression for the sequence of matrices $\{G_m\}$. In order to achieve this, we need some auxiliary matrices. We define GL_m as the matrix with $GL_m[i, j] = \langle \bar{\imath}, \bar{\jmath} \rangle_m^L$, where $\langle \bar{\imath}, \bar{\jmath} \rangle_m^L = \langle 0\bar{\imath}, 0\bar{\jmath} \rangle$ can be viewed as the number of tilings of the region bounded by consecutive paths $\bar{\imath}$ and $\bar{\jmath}$, which has been modified by adding an extra unit square on the upper left corner of the $m \times 2$ rectangle. Similarly, let GR_m be the matrix with $GR_m[i, j] = \langle \bar{\imath}, \bar{\jmath} \rangle_m^R$, where $\langle \bar{\imath}, \bar{\jmath} \rangle_m^R = \langle 1\bar{\imath}, 1\bar{\jmath} \rangle$ represents the number of possible tilings of the region bounded by $\bar{\imath}$ and $\bar{\jmath}$, which has been modified by adding an extra unit square on the upper right corner. Finally, let GT_m be the matrix with coefficients $GT_m[i, j] = \langle \bar{\imath}, \bar{\jmath} \rangle_m^T$, where $\langle \bar{\imath}, \bar{\jmath} \rangle_m^T = \langle 0\bar{\imath}, 1\bar{\jmath} \rangle$ is the number of possible tilings of the region bounded by paths $\bar{\imath}$ and $\bar{\jmath}$, which includes two extra unit squares on the top of the rectangle. To understand better the meaning of these new matrices, the reader can observe FIGURE 7 and realize that $\langle \overline{5}, \overline{31} \rangle_5^L = 1$, $\langle \overline{5}, \overline{31} \rangle_5^R = 1$ and $\langle \overline{5}, \overline{31} \rangle_5^T = 0$.

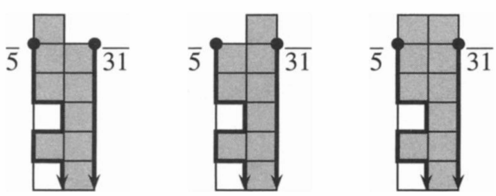


Figure 7 One extra unit square on the left, on the right, and two extra unit squares on the top.

It is not a hard task to verify that the relations listed in TABLE 1 hold; most of them are trivial. As an example, in FIGURE 8 we have $\bar{\imath} = \overline{16}$, $\bar{\jmath} = \overline{63}$, and the relation

$$\langle 0\bar{\imath}, 1\bar{\jmath} \rangle_{m+1}^T = \langle \bar{\imath}, \bar{\jmath} \rangle_m^L + \langle \bar{\imath}, \bar{\jmath} \rangle_m^R$$

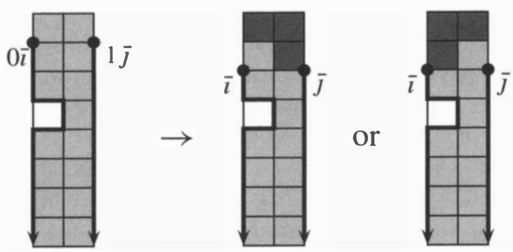
reduces to the equality $4 = 2 + 2$.

From these relations we get the following result.

THEOREM 1. *The matrices G_m, GL_m, GR_m and GT_m satisfy the following properties:*

TABLE 1: Relations among the coefficients of G_m, GL_m, GR_m, and GT_m

$\langle 0\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m^L$		
$\langle 0\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m^T$		
$\langle 1\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m$		
$\langle 1\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m^R$		
$\langle 0\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}^T$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m$		
$\langle 0\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^T$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m^L + \langle \bar{\imath}, \bar{\jmath}\rangle_m^R$		
$\langle 1\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}^T$	$=$	0		
$\langle 1\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^T$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m$		
$\langle 0\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}^L$	$=$	$\langle 0\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}^R$	$=$	0
$\langle 0\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^L$	$=$	$\langle 0\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^R$	$=$	$\langle \bar{\imath}, \bar{\jmath}\rangle_m$
$\langle 1\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}^L$	$=$	$\langle 1\bar{\imath}, 0\bar{\jmath}\rangle_{m+1}^R$	$=$	0
$\langle 1\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^L$	$=$	$\langle 1\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^R$	$=$	0



$$\langle 0\bar{\imath}, 1\bar{\jmath}\rangle_{m+1}^T = \langle \bar{\imath}, \bar{\jmath}\rangle_m^L + \langle \bar{\imath}, \bar{\jmath}\rangle_m^R$$

Figure 8 Illustrating one relation of TABLE 1.

(a) *For $m = 0$ we have*

$$G_0 = [1], \quad GL_0 = GR_0 = GT_0 = [0].$$

(b) *For $m > 0$ the following recursive relations hold:*

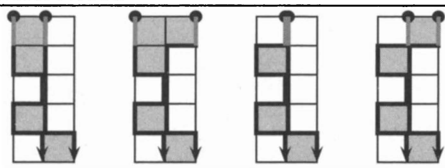
$$G_{m+1} = \begin{bmatrix} GL_m & GT_m \\ G_m & GL_m \end{bmatrix}, \qquad GR_{m+1} = GL_{m+1} = \begin{bmatrix} Z_m & G_m \\ Z_m & Z_m \end{bmatrix},$$

$$GT_{m+1} = \begin{bmatrix} G_m & 2GL_m \\ Z_m & G_m \end{bmatrix}$$

Here, Z_m represents the $2^m \times 2^m$ zero matrix.

Proof. The first part follows from the fact that G_0 relates to the number of tilings of a null-area region, while GL_0, GR_0, and GT_0 relate to nontilable regions that have respective areas of 1, 1, and 2 unit squares.

For the second part of the theorem, let us suppose that we already know the matrices G_m, GL_m, GR_m and GT_m. Let us start with G_{m+1}. We can classify all pairs of consecutive $(m+1)$-binary paths in four groups: $\{0\bar{\imath}, 0\bar{\jmath}\}$, $\{0\bar{\imath}, 1\bar{\jmath}\}$, $\{1\bar{\imath}, 0\bar{\jmath}\}$ and $\{1\bar{\imath}, 1\bar{\jmath}\}$ (see FIGURE 9). But the submatrix of G_{m+1} which corresponds to the first of these groups



$\{0\bar{\imath}, 0\bar{\jmath}\}$ $\{0\bar{\imath}, 1\bar{\jmath}\}$ $\{1\bar{\imath}, 0\bar{\jmath}\}$ $\{1\bar{\imath}, 1\bar{\jmath}\}$

Figure 9 Different pairs of $(m+1)$-binary paths in terms of their initial behaviour.

is precisely the upper left $2^m \times 2^m$ block, and by TABLE 1 this block must coincide with GL_m. In a similar way, the second group is related to the upper right $2^m \times 2^m$ block, and it has to coincide with GT_m. For the third and fourth groups we obtain respectively the lower left block, which must be G_m, and the lower right block, equal to GR_m. Therefore,

$$G_{m+1} = \begin{bmatrix} GL_m & GT_m \\ G_m & GR_m \end{bmatrix}.$$

This reasoning can be performed in a similar way, by using the relations in TABLE 1, with the matrices GL_{m+1}, GR_{m+1} and GT_{m+1}, obtaining respectively

$$GL_{m+1} = \begin{bmatrix} Z_m & G_m \\ Z_m & Z_m \end{bmatrix}, \quad GR_{m+1} = \begin{bmatrix} Z_m & G_m \\ Z_m & Z_m \end{bmatrix},$$

$$GT_{m+1} = \begin{bmatrix} G_m & GL_m + GR_m \\ Z_m & G_m \end{bmatrix}.$$

From the equality $GL_m = GR_m$ we finally obtain the stated result. ■

```
<< LinearAlgebra'MatrixManipulation'
Array[GL, 12]
Array[G, 12]
Array[GT, 12]
Array[Z, 12]
GL[0] = {{0}}
G[0] = {{1}}
GT[0] = {{0}}
Z[0] = {{0}}
Do[
  GL[i + 1] = BlockMatrix[{{Z[i], G[i]}, {Z[i], Z[i]}}];
  G[i + 1] = BlockMatrix[{{GL[i], GT[i]}, {G[i], GL[i]}}];
  GT[i + 1] = BlockMatrix[{{G[i], 2GL[i]}, {Z[i], G[i]}}];
  Z[i + 1] = BlockMatrix[{{Z[i], Z[i]}, {Z[i], Z[i]}}],
  {i, 0, 11}]
```

Figure 10 *Mathematica* instructions to compute G_m up to $m = 12$.

Applications

Enumeration and existence of tilings With a simple set of instructions in *Mathematica* (see FIGURE 10) or similar math packages we can compute the matrix G_m for the first values of m—we were able to compute G_m comfortably up to $m = 12$. By means of these calculations it is easy now to find the number of different tilings with right trominoes for some special cases:

COROLLARY 1. *For $3k \times 3k$ squares we have:*

(a) *The number of different tilings of a* 3×3 *square is* 0.
(b) *The number of different tilings of a* 6×6 *square is* 162.
(c) *The number of different tilings of a* 9×9 *square is* 1193600.
(d) *The number of different tilings of a* 12×12 *square is* 2033502499954.

Proof. Just check out the values of the coefficients $G_3^2[0, 7]$, $G_6^5[0, 63]$, $G_9^8[0, 511]$, and $G_{12}^{11}[0, 4095]$. ■

As an extension to this corollary, TABLE 2 shows the number of different tilings for rectangles whose sides are up to 9 units long.

TABLE 2: Number of tilings for small $m \times n$ rectangles

Height (m) \ Width (n)	2	3	4	5	6	7	8	9
2	0	2	0	0	4	0	0	8
3		0	4	0	8	0	16	0
4			0	0	18	0	0	88
5				0	72	0	0	384
6					162	520	1514	4312
7						0	0	22656
8							0	204184
9								1193600

Now we take a closer look at the existence of tilings in m-strips. We say that a strip is *size appropriate* if it contains a number of unit squares that is a multiple of 3—only in this case a strip can possibly be tiled with trominoes. The reader should notice that given m-binary paths $\bar{\imath}$ and $\bar{\jmath}$, whenever $m \not\equiv 0 \pmod 3$ exactly one of the m-strips $S_m(\bar{\imath}, \bar{\jmath}, n)$, $S_m(\bar{\imath}, \bar{\jmath}, n+1)$ and $S_m(\bar{\imath}, \bar{\jmath}, n+2)$ is size appropriate. On the other hand, when $m \equiv 0 \pmod 3$ the size appropriateness of an m-strip does not depend on n. As we said before, the coefficient $G_m^{n-1}[i, j]$ counts the number of different tilings that $S_m(\bar{\imath}, \bar{\jmath}, n)$ has. In case this number is zero, we can say that the strip $S_m(\bar{\imath}, \bar{\jmath}, n)$ is not tilable. Computations of the powers of G_m show us nontrivial examples of strips that cannot be tiled with right trominoes (see FIGURE 11).

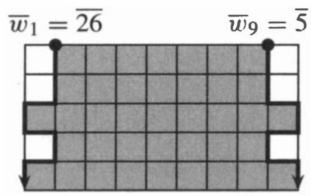


Figure 11 An example of a size appropriate strip that cannot be tiled with right trominoes.

Our strategy to obtain results on the tilability of m-strips for arbitrarily large m is based on calculations performed for m-strips with small values of m. These calculations can be managed by our mathematical software, and the results obtained from them help us confront the general case with success.

THEOREM 2. *The following statements hold:*

1. *Any size appropriate strip* $S_2(\bar{\imath}, \bar{\jmath}, n)$ *can be tiled with right trominoes.*
2. *Any size appropriate strip* $S_4(\bar{\imath}, \bar{\jmath}, n)$ *with* $n \geq 6$ *can be tiled with right trominoes.*
3. *For* m *even,* $m \geq 6$*, any size appropriate strip* $S_m(\bar{\imath}, \bar{\jmath}, n)$ *with* $n \geq 11$ *can be tiled with right trominoes.*

Proof. To prove 1, let us consider the sequence of matrices $M_{2,k} = G_2^{1+3k} + G_2^{2+3k} + G_2^{3+3k}$, $k \geq 0$, which satisfies the recursion formula $M_{2,k+1} = M_{2,k} \cdot G_2^3$. Note that all the involved matrices have nonnegative coefficients. In fact, the matrix $M_{2,0}$ has all its entries strictly positive, and this is also true for each $M_{2,k}$, $k > 0$—otherwise, G_2 and hence $M_{2,0}$ would have a row of zeroes, which is not the case. Now, let us take a size appropriate 2-strip of order n. Then n must be greater than 1 and there is some k such that $n - 1 \in \{1 + 3k, 2 + 3k, 3 + 3k\}$. If the strip is bounded by paths $\overline{w}_1$ and $\overline{w}_n$, the coefficient $[w_1, w_n]$ in $M_{2,k}$ is nonzero, and therefore, that of G_2^{n-1} has to be also nonzero, for the corresponding coefficient in each of the other two matrix terms of $M_{2,k}$ must be zero, being related to strips which are not size appropriate. This implies statement 1. A similar argument can be used to prove the second assertion, starting with the sequence of matrices $M_{4,k} = G_4^{5+3k} + G_4^{6+3k} + G_4^{7+3k}$, $k \geq 0$, and checking that the first of these matrices (and, consequently, the rest of them) has all its entries greater than zero.

To prove the last part of the theorem, let us consider a size appropriate m-strip of order $n \geq 11$, with m even and greater than 4. To show that it can be tiled, let us divide the strip into left and right regions of lengths $n_1 \geq 6$ and $n_2 \geq 6$, so that $n_1 + n_2 = n + 1$ and both regions overlap in a central $m \times 1$ rectangle. Introduce now some auxiliary segments $L_1, L_2, \ldots$ and $R_1, R_2, \ldots$ respectively in these left and right regions of the strip, as shown in FIGURE 12.

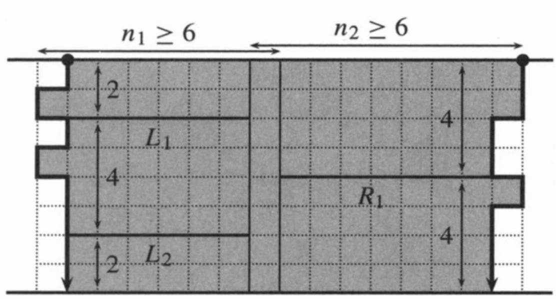


Figure 12 Tiling a size appropriate m-strip with m even.

The left segments divide the left part of the strip in substrips of heights 2, 4, 4, ..., 4/2 and the right ones in substrips of heights 4, 4, 4, ..., 2/4—the last height depending on whether $m \equiv 0 \pmod 4$ or $m \equiv 2 \pmod 4$. These substrips have an "open" border along the middle (overlapping) strip; the good thing about this configuration is that it allows us to construct an intermediate m-binary path $\overline{w}_{n_1}$ which splits the initial strip into smaller substrips, all of them being size appropriate. For, starting with the upper left substrip, we can close it so that it becomes a size appropriate 2-strip (the height of 2 units gives us freedom to achieve this by adding 0, 1, or 2 unit squares);

after closing it, we move into the upper right substrip, and close it creating a size appropriate 4-strip; then we move to the second left substrip and close it in a similar way, always producing size appropriate smaller substrips in our process. Since our initial m-strip was size appropriate, the last substrip which becomes closed without freedom to choose its open border is bound to be size appropriate as well, and therefore, by previous parts 1 and 2, all these substrips can be tiled and we obtain a complete tiling of the m-strip. ∎

For m-strips with m odd, things become a little bit trickier. Given m odd, let us give special names to the paths $\bar{l}_m = 010101\ldots10$ and $\bar{r}_m = 101010\ldots01$ (m bits); we refer to them as *left exceptional* and *right exceptional* m-binary paths respectively (see FIGURE 13).

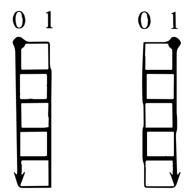


Figure 13 Left and right exceptional paths.

It is easy to realize that whenever a strip is bounded on its left by $\bar{l}_m$ or on its right by $\bar{r}_m$, then it cannot be tiled. In the corresponding matrix G_m this translates into the fact that the row l_m and the column r_m only contain zeroes. Now we have:

THEOREM 3. *The following statements hold:*

1. *Any size appropriate strip $S_5(\bar{\imath}, \bar{\jmath}, n)$ with $n \geq 10$, $\bar{\imath} \neq \bar{l}_5$ and $\bar{\jmath} \neq \bar{r}_5$ can be tiled with right trominoes.*
2. *Any size appropriate strip $S_7(\bar{\imath}, \bar{\jmath}, n)$ with $n \geq 8$, $\bar{\imath} \neq \bar{l}_7$ and $\bar{\jmath} \neq \bar{r}_7$ can be tiled with right trominoes.*
3. *For m odd, $m \geq 9$, any size appropriate strip $S_m(\bar{\imath}, \bar{\jmath}, n)$ with $n \geq 21$, $\bar{\imath} \neq \bar{l}_m$ and $\bar{\jmath} \neq \bar{r}_m$ can be tiled with right trominoes.*

Proof. We follow a parallel argument to the one used in Theorem 2, introducing the necessary variations. For part 1 let us consider the sequence of matrices $M_{5,k} = G_5^{9+3k} + G_5^{10+3k} + G_5^{11+3k}$, $k \geq 0$. This sequence is formed by nonnegative matrices and satisfies $M_{5,k+1} = M_{5,k} \cdot G_5^3$. Actually, its first term has all its entries strictly positive, except those which correspond to the row l_5 or the column r_5. This property is also shared by each matrix $M_{5,k}$, because G_5 is a matrix with nonnegative coefficients that only has one row of zeroes (l_5) and one column of zeroes (r_5)—in case some $M_{5,k}$ contained more zeroes apart from those in row l_5 and column r_5 we would deduce that $M_{5,0}$ has another row or column of zeroes. As in Theorem 2, this implies the result for 5-strips. Similarly, to prove part 2 we use the sequence of matrices $M_{7,k} = G_7^{7+3k} + G_7^{8+3k} + G_7^{9+3k}$, $k \geq 0$.

To prove part 3 we have to be more careful than in the even case. We again divide the m-strip into left and right regions, each of them of widths n_1 and n_2, with $n_1, n_2 \geq 11$ and $n_1 + n_2 = n + 1$, overlapping at an intermediate $m \times 1$ rectangle (see FIGURE 14).

Now, since the left path is not $\bar{l}_m$, we can trace two segments L_1 and L_2 which split the left region in consecutive substrips of heights even, 5, and even (one of them may have zero height!) and so that the central substrip does not have as left path a left

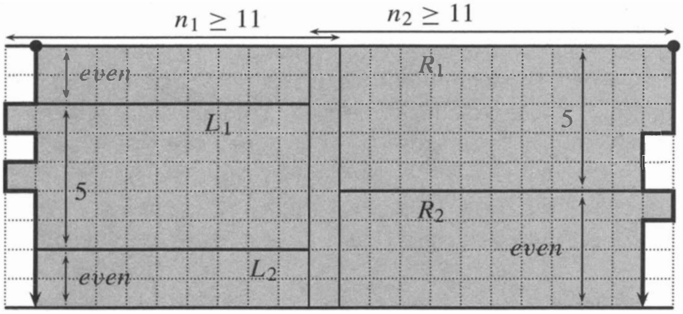


Figure 14 Tiling a size appropriate m-strip with m odd.

exceptional one. On the right region we proceed similarly, obtaining segments R_1 and R_2 which divide it in substrips of heights even, 5 and even—being careful to choose the 5-strip in such a way that it does not have a right exceptional path as right border. In some cases where there is some L_i and R_j with a difference in height of only 0 or 1 units, we have to *make room* and move upwards or downwards one or two of those segments, substituting one or both of the 5 unit height substrips by 7 unit height substrips (see FIGURE 15).

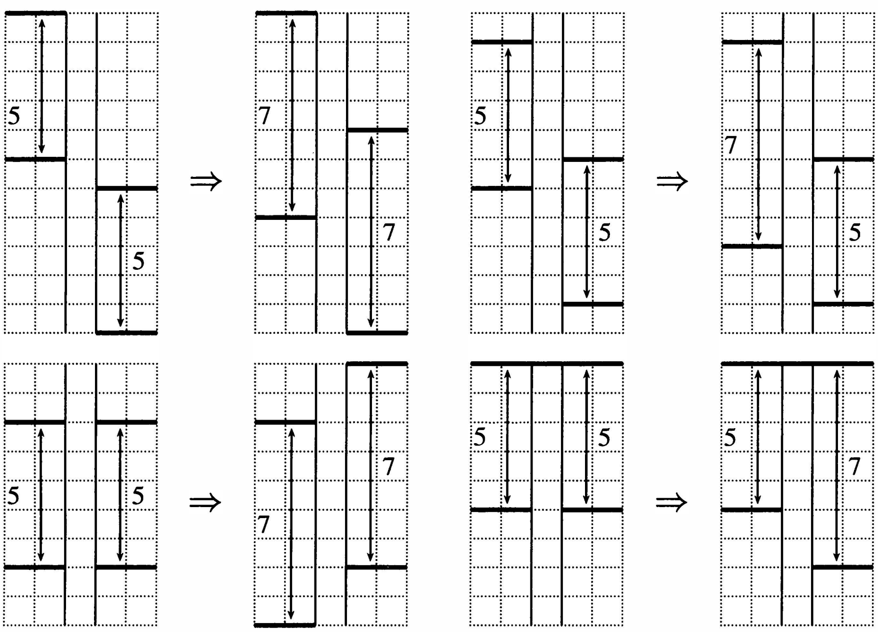


Figure 15 Turning 5-substrips into 7-substrips to make room.

After doing this, we can proceed to construct a central m-binary path $\overline{w}_{n_1}$ which is going to split the initial strip into size appropriate substrips—taking care to avoid exceptional paths when closing the substrips with odd heights—and, by the previous parts and by Theorem 2, each of them can be tiled to give us a full tiling of the initial m-strip. ■

REMARKS. The lower bound for n in the third part of theorems 2 and 3 can probably be improved significantly. On the other hand, the lower bounds considered for $m \in \{4, 5, 7\}$ are sharp. For example, FIGURE 11 shows a size appropriate 5-strip of order 9 which is not tilable. As an exercise, the reader can find examples of size appropriate, nontilable regions of maximum order for $m = 4$ and $m = 7$.

Generating functions for the number of tilings of rectangles. We define the *generating function $f_m(t)$ for the number of tilings of rectangles of height m* as the formal power series

$$f_m(t) = 1 + G_m[0, 0]t + G_m^2[0, 0]t^2 + G_m^3[0, 0]t^3 + \cdots ,$$

where the coefficient of t^n gives us the number of possible tilings of the rectangle $m \times n$. Let us consider the matrix G_m as the matrix representation of a linear automorphism in the $\mathbb{R}$-vector space $\mathbf{V}$ generated by the linear combinations with real coefficients of m-binary paths, where we work with the natural basis $B = \{\overline{0}, \overline{1}, \ldots, \overline{2^m - 1}\}$. Let us now give to $\mathbf{V}$ the structure of an $\mathbb{R}[t]$-module by setting $t\vec{v} = G_m(\vec{v})$, $\vec{v} \in \mathbf{V}$. If we set $\vec{w} = \overline{0}$, then the first coordinate of the vector $G_m^n(\vec{w})$ is equal to $G_m^n[0, 0]$, the number of possible tilings of an $m \times n$ rectangle. We are going to center our attention in the annihilator ideal $\text{Ann}(\vec{w})$ of $\mathbb{R}[t]$. If a polynomial $p_m(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$ belongs to this ideal—as the characteristic polynomial $\chi_m(t)$ of G_m does—then

$$(G_m^k + a_{k-1}G_m^{k-1} + \cdots + a_1 G_m + a_0 I)\vec{w} = \vec{0},$$

and, in general,

$$(G_m^{k+i} + a_{k-1}G_m^{k+i-1} + \cdots + a_1 G_m^{i+1} + a_0 G_m^i)\vec{w} = \vec{0}, \quad i \in \mathbb{N}.$$

From this we obtain

$$G_m^{k+i}[0, 0] + a_{k-1}G_m^{k+i-1}[0, 0] + \cdots + a_1 G_m^{i+1}[0, 0] + a_0 G_m^i[0, 0] = 0.$$

This recursive relation allows us to construct a rational function that, when expanded as a power series at $t = 0$, coincides with $f_m(t)$ (see [**7**, Chapter 1]). Again, some computer code can give us such rational expression for the functions $f_m(t)$ for the first values of m—compare to [**5**]. This expression becomes quickly quite large, as TABLE 3 shows.

Concluding remarks

Some possible directions for further exploration in this area are the following:

Tilings with other polyominoes. Is it possible to use a similar method to that used to prove Theorem 1 with other types of tilings by polyominoes, without getting into excessive difficulties?

Properties of the sequence $\{G_m\}$. What else can be said about these matrices? It would be particularly interesting to understand better the sequence $\{\rho(G_m)\}$ of spectral radii—the spectral radius $\rho(A)$ of a matrix A is the maximum of the moduli of its eigenvalues—for they are closely related to the number of tilings of $m \times n$ rect-

TABLE 3: Characteristic polynomials and generating functions for $2 \le m \le 6$

$$\chi_2(t) = t(t^3 - 2)$$

$$f_2(t) = \frac{1}{1 - 2t^3}$$

$$\chi_3(t) = t^2(t-1)^2(t+1)^2(t^2-2)$$

$$f_3(t) = \frac{1}{1 - 2t^2}$$

$$\chi_4(t) = t^4(t^3 - 2)(t^9 - 10t^6 + 22t^3 + 4)$$

$$f_4(t) = \frac{1 - 6t^3}{1 - 10t^3 + 22t^6 + 4t^9}$$

$$\chi_5(t) = t^{14}(t^6 + 2t^3 + 5)(t^{12} - 2t^9 - 103t^6 - 280t^3 - 380)$$

$$f_5(t) = \frac{1 - 2t^3 - 31t^6 - 40t^9 - 20\,t^{12}}{1 - 2t^3 - 103t^6 - 280t^9 - 380t^{12}}$$

$$\begin{aligned}\chi_6(t) = t^{18}&(t-1)^2(t+1)(t^5 + t^4 - 3t^3 - 9t^2 - 2t + 8)^2\\ &(t^6 + t^5 - t^4 - 5t^3 - 2t^2 + 4t - 2)\\ &(t^8 - 6t^6 - 18t^5 + 3t^4 + 42t^3 + 50t^2 - 4t - 32)^2\\ &(t^{11} - 2t^{10} - 8t^9 - 2t^8 + 43t^7 + 42t^6 - 36t^5 - 102t^4 + 44t^2 + 8t + 8)\end{aligned}$$

$$f_6(t) = \frac{1 - 2t - 4t^2 - 2t^3 + 13t^4 + 6t^5 - 6t^6 - 6t^7}{1 - 2t - 8t^2 - 2t^3 + 43t^4 + 42t^5 - 36t^6 - 102t^7 + 44t^9 + 8t^{10} + 8t^{11}}$$

angles as n increases indefinitely. Computer calculations give us the first values of this sequence:

$$\begin{aligned} \rho(G_2) &= 1.25992 & \rho(G_7) &= 4.06693\\ \rho(G_3) &= 1.41421 & \rho(G_8) &= 5.38729\\ \rho(G_4) &= 1.87061 & \rho(G_9) &= 7.09995\\ \rho(G_5) &= 2.31233 & \rho(G_{10}) &= 9.36233\\ \rho(G_6) &= 3.15986 & \rho(G_{11}) &= 12.3453 \end{aligned}$$

Acknowledgment. The author would like to thank the referees for the careful reading of this article, as well as for the various suggestions and corrections that notably improved its final version.

REFERENCES

1. S.W. Golomb, *Polyominoes: Puzzles, Patterns, Problems and Packings*, Revised edition, Princeton Univ. Pr., 1994.
2. S.W. Golomb and J. Marshall Ash, *Tiling deficient rectangles with trominoes*, this MAGAZINE **77** (2004) 46–55.
3. S.W. Golomb, Checker boards and polyominoes, *Amer. Math. Monthly* **61** (1954) 675–682.
4. I-P. Chu and R. Johnsonbaugh, *Tiling deficient boards with trominoes*, this MAGAZINE **59** (1986) 34–40.
5. Cristopher Moore, Some polyomino tilings of the Plane, e-print at `arXiv:math/9905012v1`, 1999.
6. S. Heubach, P. Chinn, and P. Callahan, Tiling with trominoes, *Congressus Numerantium* **177** (2005) 33–44.
7. Herbert S. Wilf, *Generatingfunctionology*, Academic Press, 1990.

The Mystery of Robert Adrain

FRANK J. SWETZ
Pennsylvania State University (Emeritus)
Middleton, PA
FJS2@PSU.edu

Questions

I first encountered the name "Robert Adrain" on the cover of an early American mathematics text, *A Course of Mathematics in Two Volumes for the Use of Academics as well as Private Tuition* written by Charles Hutton (1737–1823) [**21**]. Hutton was a self educated British mathematician noted for his textbook writing, particularly books dealing with "practical mathematics" [**20**]. As an instructor at the Royal Military Academy in Woolrich, England, he originally compiled the work in three volumes during the years 1798–1801 for use by his cadets. Hutton's series was a compendium of the mathematics deemed necessary for a nineteenth century military career. Its scope went from basic arithmetic to the applications of calculus and stressed utility in such fields as statistics, dynamics, the theory of projectiles and hydrology. When the United States Military Academy formally opened at West Point in 1801, the study of this book was included in its syllabus. Its contents became the basis for the first mathematics course taught at the new academy [**4**]. Adrain edited, revised, and condensed the British text and in 1812 published an American version consisting of two volumes. The American edition underwent four reprints and the British edition thirteen and remained in use at West Point until 1823. Indeed, Hutton's *Mathematics* was a very popular and useful book in its time. But 'Who was Robert Adrain, was he British or American?' Although I was familiar with some early American mathematicians and textbook writers such as Bowditch, Greenwood, Pike, and Winthrop, the name Adrain eluded me. A small citation on the front cover of the text identified Adrain as Professor of Mathematics and Natural Philosophy at Columbia College in New York City and a Fellow of the American Philosophical Society. Robert Adrain apparently was an American! I moved on to examine the contents of *A Course of Mathematics*, attempting to ascertain its mathematical relevance to a newly founded nation.

A year later I undertook a similar task in examining the contents of the *Mathematical Correspondent*, the first mathematics journal published in the United States of America [**39**]. This journal was founded and initially edited by George Baron (1769–1812), a contentious mathematician, who very briefly (1801–1802) served at West Point as the first civilian "Teacher of the Arts and Sciences to the Artillerists and Engineers" [**18**]. A quarterly publication, the *Correspondent* attempted to emulate the successful and influential *Ladies Diary*, a British periodical edited by Charles Hutton and devoted to problem solving. The *Diary* helped to popularize mathematics in eighteenth century England [**28**]. Baron felt that a similar effort was warranted in the United States to advance mathematical knowledge and help to form a mathematics community. This new American journal appeared in May of 1804 and was mainly comprised of problems and their eventual solutions as posed and posted by subscribers. Occasionally, it would include an essay expounding and explaining a selected topic in mathematics but it was mainly a problem solving journal as were most contemporary mathematical periodicals. Problem solving was believed to provide a "key" for mathematical understanding. In order to promote competition, Baron offered a $6 prize, a decent sum at this time, for the "best" correct solution offered for a submitted problem judged most difficult or complex, the "prize problem". Problems involved such subjects as: monetary exchange; navigation; commercial business transactions; and land surveying and reflected the mathematical life and needs of nineteenth century America [**38**]. As a further inducement to readers, the names of all correct problem solvers were listed at the end of each edition. David Zitarelli in his examination of the *Correspondent* has singled out such listings as a valuable research tool:

> Overall, the list of contributors provides a priceless, 200-year-old portrait of America's first [mathematics] publication community, supplying a glimpse of the initial stage of what would develop into a legitimate community of research mathematicians a hundred years later [**39**, p. 8].

Among this listing, the name Robert Adrain stood out as the most prolific problem solver. In the short life of the *Mathematical Correspondent* (it only ran for 9 issues) Adrain submitted seven problems and solved eighty-nine, including fifteen prize problems. Twice his solution for the "prize problem" claimed the reward. He remained unique in this accomplishment. Further, he contributed two articles to the journal: "Disquisition concerning the Motion of a Ship which is steered to a certain Point of the Compass" where he discusses the effects of the earth's rotation on a moving ship [(1807), p. 103] and "View of Diophantine Algebra" [(1807), pp. 212–241; (1808), pp. 7–17]. In the latter article Adrain discusses the solution of Diophantine equations and solves several specific problems illustrating solution techniques for these equations. His instructions on this topic would continue in the second edition of the journal (1807). This article on Diophantine analysis was the first on the subject published in the United States. Obviously Robert Adrain was more than just a mathematical gadabout. He was certainly an avid problem solver but, mathematically speaking, was he much more? A quick internet search: i.e., MacTutor History of Mathematics website [**25**], revealed that, indeed, Robert Adrain was a recognized member of the early American mathematical community. Further, several published articles have examined the man's life and work in some detail, notably: Julian Coolidge's "Robert Adrain and the Beginnings of American Mathematics", which was the text of his 1925 retirement address as President of the Mathematical Association of America [**10**], and Edward Hogan's "Robert Adrain: American Mathematician" [**19**]. As a teacher, an accomplished applied mathematician, a developer of curriculum, an editor, a writer, and an evangelist of mathematics, he was apparently a prime mover in early nineteenth century mathematics education but this particular distinction seems to remain unrecognized. A mystery remained.

Adrain: The man and his career

Robert Adrain was born September 30, 1775 in Carrickfergus, Ireland. His father was a school master and maker of mathematical instruments. Robert's precocious intellect was recognized at an early age and his father set him on a classical education intended for the ministry. When he was fifteen years old, his parents died and Robert had to terminate his formal education to support himself and his four brothers and sisters. He assumed his father's vacant position as a teacher. Prospering in his new career, he expanded his knowledge and developed an interest in mathematics which he pursued through diligent self-study. Mathematics and its power fascinated him. In 1798 Adrain married and also participated in the ill-fated Irish Rebellion of that year. The Rebellion left him a fugitive with a price of £50 on his head and he fled with his wife and a child to America. Landing in New York City during a cholera epidemic, the family sought refuge in Princeton, New Jersey where acquaintances and the promise of a job awaited. Robert Adrain briefly served as a Mathematics Master at Princeton Academy before moving in 1800 to York, Pennsylvania to assume the Head-Mastership of the York Academy. It was during his tenure at York that he began contributing to the *Correspondent*. In 1805 he and his family moved to Reading, Pennsylvania where he became the principal of its academy [**29**]. In 1807 Baron was dying of consumption and gave up the editorship of the *Correspondent*. Adrain then became editor of the faltering journal and attempted to revive it. He failed and within a year it ceased publication [**14**].

In 1808 he began his own journal, *The Analyst or Mathematical Museum*, fashioned after the *Correspondent*, but focused at a higher level of mathematical involvement both in problem solving and exposition. The cover of the first issue described its contents with the same words used to depict the *Correspondent*:

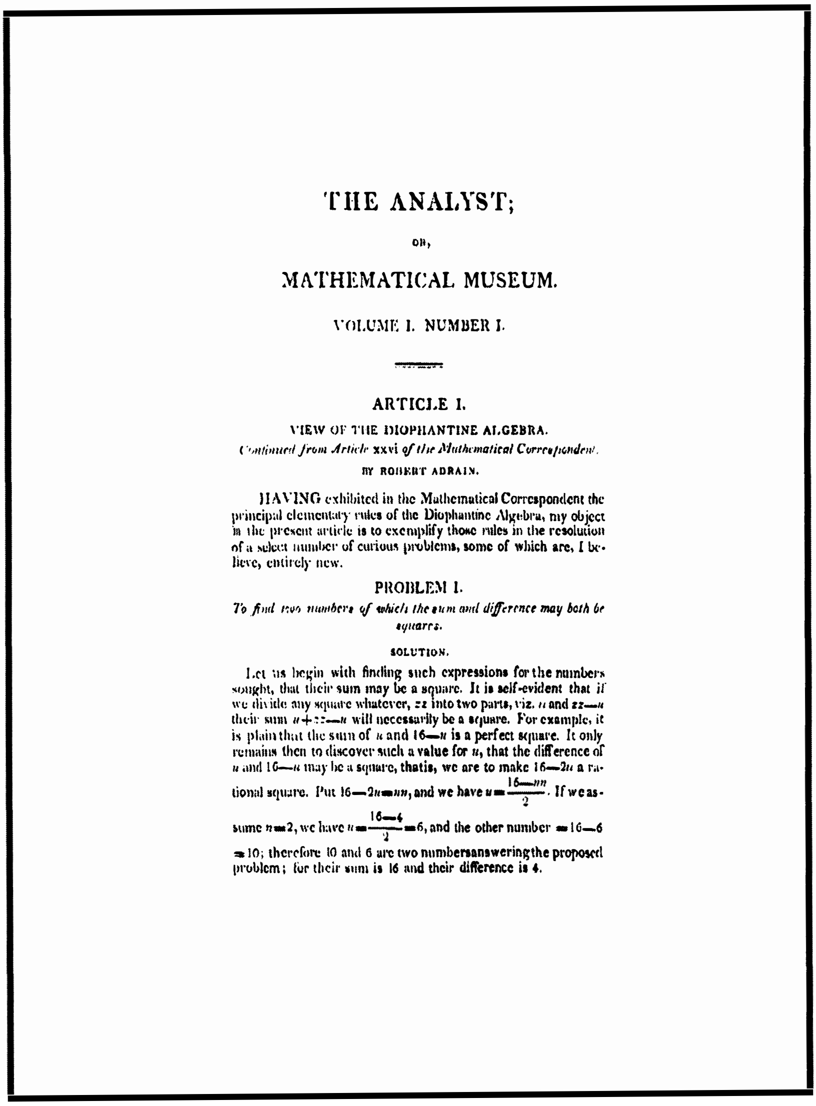
> Containing new elucidations, improvements, and
> discoveries, in the various branches of the mathematics;
> with selections of new and interesting questions,
> proposed and resolved by ingenious correspondents.

Although printers and the location of publication varied, this journal continued under Adrain's editorship until 1814 when, due to a lack of subscribers, it also ceased functioning.

In 1809 Adrain was appointed the first Professor of Mathematics at Queens College in New Brunswick, New Jersey. He retained this position until 1813 when he was hired as a Professor by Columbia College in New York City. While at Columbia, serving as Professor of Mathematics and Physics (1813–1820) and Professor of Mathematics and Astronomy (1820–1825), he also contributed mathematical material to the *Portico* (1816–1820); *The Scientific Journal* (1818–1819); *The Ladies' and Gentlemen's Diary* (1819–1821); and the weekly, *The New York Mirror and Ladies' Literary Gazette* (1823–1826) where, through his writing, he would "tend to promote the invaluable science of mathematics" [(1823), 1:3]. In 1825 Adrain initiated a new journal, *The Mathematical Diary*, which he edited for a year before returning to teach again at Queens College. However he continued supporting and writing for this journal until its eventual demise in 1832.

It was during his transition to Columbia that his edited version of Hutton's *Mathematics* appeared. Besides condensing the material, Adrain reorganized it and corrected several mistakes, specifically: on the reduction of fractions; application of logarithms; definition of surds and improved geodetic estimates. The reprinting of Hutton's book in 1822 contained an essay by Adrain on elementary descriptive geometry [vol. 2, pp. 561–622]. Gaspard Monge's *Géometrie Descriptive* had been published in 1799. The first American appearance of this subject was Claude Crozet's, *A Treatise on Descriptive Geometry for the Use of Cadets of the United States Military Academy* (1821) [**23**, pp. 239–240] but this edition was for a limited audience. Crozet, a former military engineer for Napoleon, had been recruited by the Academy to impart an École Polytechnique flavor to its teaching. Adrain's "essay" was the first popular exposition on this subject in the United States and appeared to be an independent work. When James Ryan's *An Elementary Treatise on Algebra* appeared in 1824, it contained an appendix written by Adrain, "Obtaining an Algebraic Method of Demonstrating the Proposition in the Fifth Book of Euclid's Elements" [**30**]. This appendix was extracted from an issue of the *Analyst* [1814, pp. 1–20].

In 1827 Robert Adrain became Professor of Mathematics at the University of Pennsylvania and also assumed the administrative post of Vice Provost of the University.



THE ANALYST;

OR,

MATHEMATICAL MUSEUM.

VOLUME I. NUMBER I.

ARTICLE I.

VIEW OF THE DIOPHANTINE ALGEBRA.

Continued from Article xxvi *of the Mathematical Correspondent.*

BY ROBERT ADRAIN.

HAVING exhibited in the Mathematical Correspondent the principal elementary rules of the Diophantine Algebra, my object in the present article is to exemplify those rules in the resolution of a select number of curious problems, some of which are, I believe, entirely new.

PROBLEM I.

To find two numbers of which the sum and difference may both be squares.

SOLUTION.

Let us begin with finding such expressions for the numbers sought, that their sum may be a square. It is self-evident that if we divide any square whatever, zz into two parts, viz. u and $zz-u$ their sum $u+zz-u$ will necessarily be a square. For example, it is plain that the sum of u and $16-u$ is a perfect square. It only remains then to discover such a value for u, that the difference of u and $16-u$ may be a square, that is, we are to make $16-2u$ a rational square. Put $16-2u=nn$, and we have $u=\frac{16-nn}{2}$. If we assume $n=2$, we have $u=\frac{16-4}{2}=6$, and the other number $=16-6=10$; therefore 10 and 6 are two numbers answering the proposed problem; for their sum is 16 and their difference is 4.

Figure 1 Adrain's article in 1808 *Analyst.*

He resigned from the University in 1834, briefly returned to private teaching, and died at the family home in New Brunswick, New Jersey in August of 1843 [**2**].

Adrain: The mathematician

As a mathematician and a natural philosopher, Robert Adrian was a multi-faceted scientist whose many interests dispersed his talent in varied directions. Today he would be considered an applied mathematician reflecting his sentiments that "The last and highest department of mathematical science consists in its applications to the laws and phenomena of the natural world." Among his fields of interests which included physics, astronomy, and geography, a paramount concern was dynamic geodesy. Specifically, many of his mathematical investigations focused on the shape of the earth. Isaac Newton's theories of universal gravitational and planetary motion as derived in the *Principia* (1687) had challenged classical models of the earth's sphericity. If planetary attractions and interactions were the driving force of a moving earth, then its shape would be an oblate spheroid, flattened at the poles and bulging at the equator. However, in contrast, Descartes' flux and vortex theory of planetary behavior, *Principia Philosophiae* (1644), also popular at this time, warranted an earth elongated at the poles. In the early part of the eighteenth century, this controversy of Newtonianism versus Cartesianism waged in the intellectual circles of Europe. At times the term "Cartesianism" was replaced with the term "Cassinianism" as the astronomer, Jacques Cassini, had published his theories in 1720 supporting the prolate spheroid concept of Descartes. Expeditions sent out in 1736 and 1737 by the Observatoire Royal to obtain accurate measures of the Earth's latitude near the equator (Peru) and near a pole (Lapland) determined that the shape of the earth conformed to the Newtonian model. A quest for more exacting mathematical descriptions for the curvature of the earth now attracted some of the greatest mathematical minds of the mid and later parts of the eighteenth century. Maclaurin in his *A Treatise of Fluxions* (1742) supported the theory of an oblate spheroid. Clairaut's *Théorie de la figure de la terre* in 1743 geometrically modeled the earth as a rotating fluid, homogeneous, spheroid. Jean d'Alembert worked out methods for spheroid attraction in *Recherches sur differents points importans du systeme du monde* (1754, 1756). This theory was further refined by the appearance of Laplace's *Traite de mécanique céleste* (1799). Adrain read French and was familiar with these works.

Mathematicians now worked to get a better fix on the shape of the earth. On the basis of fifteen pendulum observations, Laplace calculated that an ellipsoid shaped earth would have an ellipticity of 1/336. Using Laplace's observations and compensating for error by employing his methods of least squares, Robert Adrain obtained a more accurate value of 1/319. He published this finding in 1818 in the *Transactions of the American Philosophical Society* [**3**]. When, in 1832, Nathaniel Bowditch (1793–1838) published the second volume of Laplace's *Mécanique céleste* in translation, he judicially selected a subset of 48 measurements out of an available 52 to apply Adrain's method of least squares and obtained a value of ellipticity of 1/297, as compared with Laplace's new published estimate of 1/230. Modern measurements have confirmed Bowditch's value. Adrain's estimate was more accurate than those offered by Laplace. Further, in his edited edition of Hutton's *Mathematics*, Adrain corrected the given value for the diameter of earth at the equator, decreasing Hutton's estimate of 7957.75 miles to 7918.7 miles. Adrain's diameter was just 7.71 miles short, deficient by less than 0.1%, of a modern, satellite obtained measurement.

In many of his published questions, he seemed to favor queries that concerned the shape of the earth, for example, from *The Analyst*:

- Which is further from the center of the earth, the mouth of the Mississippi River or its source?
  Adrain's answer: the mouth is two miles farther than the source [(1814), p. 24].
- What figure will a perfectly elastic hoop take if it is acted on by two equal and opposite forces at the extremities of a diameter [(1808), p. 69]?
- What surface will such a hoop assume if of uniform strength, thickness and density when revolving with uniform angle of velocity in free and non-gravitating space [(1808), p. 111]?
- To determine the nature of the *catenaria volvens*, or the figure which a perfectly flexible chain of uniform density and thickness will assume, when it revolves with a

constant angular velocity about an axis, to which it is fastened at its extremities, in free and non-gravitating space (i.e., Catenary of revolution) [(1808), p. 72].

Adrain worked out a solution for this last problem which led to elliptic integrals of a form that would not be solved until 1860 by R. F. Clebsch at Göttingen. Historically, this period provided a fertile climate for an understanding of elliptic integrals. In Europe such mathematical notables as: Legendre, Gauss, Abel, and Jacobi investigated their properties and sought solutions. In 1840 Joseph Liouville proved the integrals non-elementary in nature. Perhaps the last significant editing Adrain did was to publish a revised, corrected, and annotated edition of Thomas Keith's *A New Treatise on the Use of Globes, or a Philosophical View of the Earth and Heavens* in 1832. In this work he challenged Keith's claim that the Andes were the highest mountains in the world. Adrain rightfully suggested that this characteristic belonged to the Himalayas. His claim was eventually confirmed by the Great Trigonometric Survey of India (1802–1860). During this survey, the height of a mountain indicated merely as "Peak 15" on British topological maps of India was determined to stand at 8850 meters—making it the highest mountain in the world! In 1865 this peak was formally named "Mt. Everest" in honor of Sir George Everest, British Surveyor General of India and Director of the survey.

Adrain's eventual fame did not result from his work in geodesics but rather from the solving of a particular *Analyst* "Prize Problem," a $10 problem set by Robert Patterson (1743–1824), Mathematics Professor at the University of Pennsylvania, consultant to the Lewis and Clark expedition, and writer of popular mathematics books:

A polygonal piece of land is measured by means of a surveyor's chain and a circumferentor, a sighting device marking bearings, thus its sides are determined

1. 40	perches	N 45° E
2. 25	perches	S 30° W
3. 30	perches	S 5° E
4. 29.6	perches	W
5. 31	perches	N 20° E

It is found that due to errors in the measurements, the polygon does not close, that is, the terminal point does not coincide with the final point. How can the polygon be adjusted as to insure closure in the best manner [(1808), p. 42]?

This issue of measurement closure was, and still is in many places in the world, a common surveying problem. The survey was conducted using a circumferentor, basically a directional compass that allows for taking bearings in a plane and a Surveyor's Chain, a Gunter's Chain, an actual metal chain of 100 links comprising100 yards or 4 standard perches, 16.5 feet each [**24**]. The prize was claimed by Nathaniel Bowditch (1773–1838) recognized mathematician, astronomer, and surveyor. Bowditch undertook the problem under two assumptions: (1) error in the length of lines would be directly proportional to length and (2) errors in bearings were equal at each sighting. He laid out the region graphically and determined the actual closure error. Resolving the error into vertical and horizontal components: 0.10 perches, south; 0.08 perch, west, he then averaged his adjustment over the vertices; triangulated the region and obtained an area 854.56 square perches.

Of course, there are numerous ways to adjust for the error, but a solution is sought which systematically adjusts by placing the vertices into their most probable positions. In two supplementary solutions that he supplied, Adrain commented on and general-

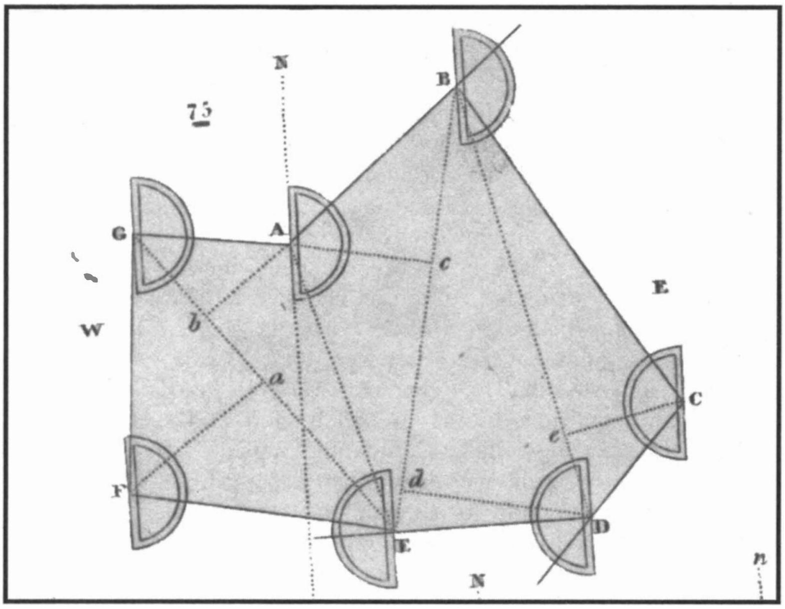


Figure 2 Illustration of the technique of chaining and triangulating a polygonal region *A, B, C, D, E, F* of land from *A Treatise on Surveying* by John Gummere (1841).

ized the problem by considering it a purely geometric situation where the measuring of several lengths was required. Using Bowditch's assumptions, he approached the problem in a probabilistic manner and derived a probability distribution for the error. Employing modern notation, the following is an outline of Adrain's derivation strategy.

In measuring two successive lengths, AB and BC, along a line where $m(AB) = \alpha$ and $m(BC) = \beta$, assume respective errors x_1, x_2 are made where $x_1 + x_2 = k$, a constant.

Under Bowditch's assumption (1),

$$\frac{x_1}{\alpha} = \frac{x_2}{\beta}.$$

Now, let the probability of making error x_1 be given by the function $P(x_1, \alpha)$ and x_2 by $P(x_2, \beta)$. Under the assumption these events are independent, the joint probability of the two events occurring is:

$$P(x_1, \alpha)P(x_2, \beta).$$

Differentiating to maximize this function, we obtain:

$$\frac{P'(x_1, \alpha)}{P(x_1, \alpha)} dx_1 + \frac{P'(x_2, \beta)}{P(x_2, \beta)} dx_2 = 0.$$

But since $dx_1 + dx_2 = 0$,

$$\frac{P'(x_1, \alpha)}{P(x_1, \alpha)} = \frac{P'(x_2, \beta)}{P(x_2, \beta)} \quad \text{when} \quad \frac{x_1}{\alpha} = \frac{x_2}{\beta}.$$

Adrain now seeks to solve this problem in "the simplest manner possible" and assumes:

$$\frac{P'(x_1,\alpha)}{P(x_1,\alpha)} = \frac{mx_1}{\alpha} \quad \text{thus}$$

$$P(x_1,\alpha) = C\exp\left(m\frac{x_1^2}{2\alpha}\right).$$

Since a maximum value is sought for $P(x_1, \alpha)$, m must be negative.

When a series of several independent errors occur, their joint probability density has the function:

$$\prod_{i=1}^{n} P(x_i,\alpha) = C\exp\left(\left[\frac{-m}{2}\right]\sum_{i=1}^{n}\frac{x_i^2}{\alpha_i}\right)$$

this will be maximized when $\sum_{i=1}^{n}\frac{x_i^2}{\alpha_i}$ is minimized, thus the "Method of Least Squares" (MLS). For more critical mathematical discussions of this derivation see [**12**, p. 177–78], [**11**, p. 67–68], and particularly [**37**, pp. 588–594]. As was his frequent practice, Adrain then set about to derive another proof for the existence of the normal distribution of error. Under the different assumption that the measures of length and bearing were independent, he represented them as rectangular coordinates, imposed geometric constraints that insured a symmetric distribution of error around a sighting point, formulated a joint probability function and maximized it as before to obtain a corresponding probability density for a measurement error x:

$$u(x) = Q\exp(-nx^2/2) \qquad Q \text{ and } n \text{ constants, determined by initial conditions}$$

A Method of Least Squares follows by the same argument as used in the previous proof. In this derivation several of Adrain's assumptions appear strained and the proof is weaker than its predecessor [**12**, p. 178]. To further illustrate and justify the method Adrain now supplies four practical applications: to determine a point on a line from varied observations; the arithmetic mean of the observations is found; to do the same for a point in space; establishing the center of gravity of the system; to correct errors of dead-reckoning at sea and to solve the surveying problem of Patterson [(1808), pp. 93–109].

Similar problems in geodesy and astronomy had also prompted Gauss and Legendre to use a Method of Least Squares. Legendre demonstrated the technique in his *Nouvelles methods pour la determination des orbites des comites* (1805). Gauss introduced the method in *Theoria Motus Corporum Coelestium* (1809), but claimed he had known of it as early as 1795. Thus Adrain's priority does not rest in devising MLS but rather in deducing a general law for the normal distribution of errors and from that law obtaining a least squares procedure. Coolidge, in his 1926 survey of Adrain's work, seems inclined to consider this accomplishment the first real mathematical discovery made in America [**11**, p. 75]. Although Adrain's method was used by Bowditch and adopted in two texts of the time: Bowditch's *The New American Navigator*, 3rd ed. (1811) and Gummere's *Treatise on Surveying* (1817), unfortunately, it received little further attention from the contemporary mathematical community and, in effect, remained forgotten for sixty years until Cleveland Abbe recalled the accomplishment in an 1871 article [**1**]. In the interim, J. F. W. Herschel duplicated the second of Adrain's proofs in an 1850 article [**17**]. It remains known as "Herschel's Proof." Since that time several researchers who have examined Robert Adrain's work on the distribution of errors in detail have concluded that it was an original and important contribution to mathematics and, indeed, the first mathematical discovery emanating from the new country of the United States of America [**7**, p. 68], [**21**, p. 581], [**11**, p.75; 31].

Other questions posed and solved by Adrain over the years indicate that he was also knowledgeable in isoperimetrics and the calculus of variation, fields of mathematics that were attracting attention in the European scientific community. For Adrain, the self-educated mathematician, working basically alone without the support and encouragement of fellow mathematicians, the scope and depth of his mathematical accomplishments are impressive.

Adrain: The teacher, educator

In attempting to understand Robert Adrain and his work, it must be remembered that he was primarily a teacher. His teaching career was demanding and varied. As a Headmaster of two early academies, i.e., York and Reading, Pennsylvania and as a Professor of Mathematics/Natural Science at three fledging universities: Queens College, later to become Rutgers University, Columbia University, and the University of Pennsylvania, he was deeply involved in building departments, establishing curriculum and setting standards, which were apparently very demanding. He could not abide any student who did not "know" his Euclid. To "know" Euclid, at this time, meant more than just being able to apply geometry, but rather to recite from memory theorems, propositions, and proofs by their assigned number. While Adrain appears as something of a "Dickinsonian" schoolmaster with rod in one hand and textbook in the other, he was also described as a kind and patient teacher, who would gladly tutor students who sought him out for assistance. Cajori commented on Adrain's "most happy facility of imparting instruction" and described him as "the most prominent teacher of mathematics" [**7**, p. 67] of the period. He was referred to as "Old Bobbie" by his students at Columbia. These same students in 1822 presented him with a portrait painted by Charles Cromwell Ingham as a testimony of their gratitude and respect, however, in later years at this same university he experienced difficulties in controlling his classes and resigned his position. His memory was failing and he no longer had a facility with foreign languages. He then returned to private tutoring and teaching at a grammar school until his death. Thus he remained a teacher of mathematics throughout his life.

In his editing and founding of mathematics journals and pamphlets and participation in mathematics discussion groups, he was constantly reaching out to a larger population promoting the applications of mathematics and the techniques of problem solving. By generalizing problems and personally demonstrating that there were often several ways to solve a particular problem, Adrain, in his published work, was encouraging a wide range of mathematical exploration. Through such examples, he was actually teaching problem solving. His *Analyst* set high standards for mathematical exposition and this journal solicited contributions from the best American mathematicians of this period drawing them together as a scientific community [**15**]. The revision and editing of Hutton's *Mathematics* was undertaken because he considered it "one of the best systems of mathematics in the English language stressing the most necessary and useful arts" [Preface, xi] relevant to the needs of the new nation. Adrain's exposition on Dioplantine algebra and descriptive geometry brought new knowledge to his reading public. His *Mathematical Diary* was the first American mathematics journal to include reviews of mathematical publications including some that appeared in Europe. It also saw the first published paper of Benjamin Pierce, who was then a student at Harvard but who would go on to become America's first native born mathematician of international recognition. Other college students followed Pierce's example by contributing to the *Diary*. Robert Adrain was an advocate of the efficacy of continental mathematical notation and helped promote it. In his work he used the differential system of Leibniz rather than the fluxions of Newton. It has been reported that while at Columbia he

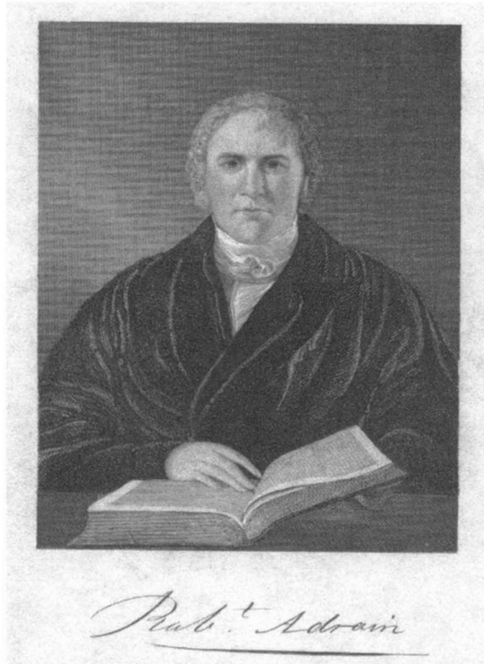


Figure 3 Portrait of Adrain presented by his class at Columbia, 1822.

wrote a textbook on calculus but ultimately becoming dissatisfied with his results, he destroyed the manuscript [**2**]. It seems strange that despite all his publications he was extremely hesitant to publish unless he was completely satisfied with his results. He was a rigorous taskmaster, even with himself. Upon his death he left a large collection of papers and research notes, the study of which was undertaken by M. J. Babb of the University of Pennsylvania. Upon Babb's death in 1945, the papers were inadvertently destroyed, leaving many unanswered questions.

How should history remember Robert Adrain?

Only two popular history of mathematics texts mention Adrain's work with the distribution of errors and MLS, Cooke and Suzuki [**10**, p. 402]; [**37**, pp. 589–594] and official surveys of the history of mathematics education in the United States ignore his accomplishments completely [**22, 32**]. Certainly during his time, he was recognized as a premier mathematician in America. Held in high esteem by his colleagues, he was awarded honorary degrees by Queen's College, an MA in 1810, and an LLD by Columbia in 1818. Adrain was elected a Fellow of the American Philosophical Society in 1813 and a year latter obtained membership in the American Academy of Arts and Sciences. But still his legacy seems clouded.

First, Adrain as a theoretical mathematician in the United States at the beginning of the nineteenth century was a man ahead of his time. From his published problems and work it appears that his ability in mathematics exceeded that of his peers who were skilled practitioners but not theorists. His interests were broader and his curiosity keener than those around him. While the majority of mathematicians in the new nation adhered to British mathematical nomenclature and models, Adrain was more in tune with continental accomplishments, particularly those of the French. As a result of these junctures, his professional interactions supplied little support or momentum for his work. Further his constant call to administrative duties also limited his research efforts. Nor did the academic climate of his university teaching provide mathematical stimulus. In general, he lived and worked in a society that held mathematics suspect and viewed it with "gentlemanly distain" [**9**]. One may wonder how Robert Adrain's mathematical career would have flourished under different circumstances; particularly had he lived and worked in Europe [**16**].

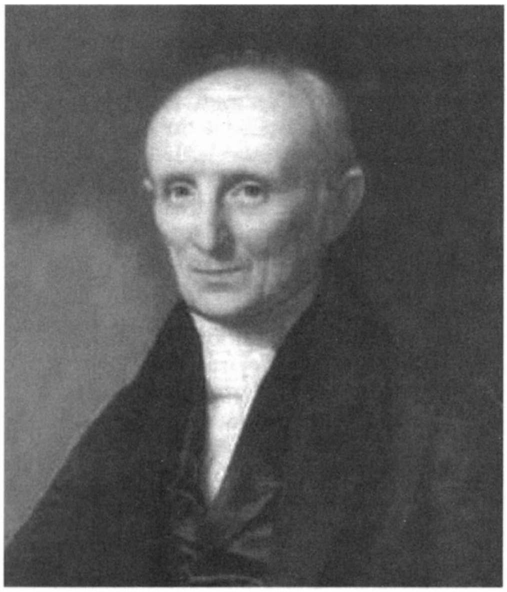


Figure 4 Nathaniel Bowditch, whose reputation overshadowed that of Robert Adrain.

Adrain's journal editorship was also fraught with frustrations. By the time Baron relinquished his charge of the *Correspondent* to Adrain, he had alienated much of its readership by his caustic comments and personal attacks on the scientific community [**15**]. Subscriptions were failing and the journal was basically defunct. Despite a sincere plea by Adrain that his tenure would be different and accommodating:

> The editor begs leave to assure the friends of science and of man, that nothing unbecoming a Christian and a gentleman shall be suffered to make its appearance in the work as long as it shall be under his direction. No affected superiority shall be shewn, nor contemptuous treatment of such as differ from us in opinion, or fall into errors [(1807), preface, vi].

The journal still faltered. In his *Analyst* endeavor, he strove to supply a more advanced approach to mathematical thinking but probably exceeded the ability of much of his audience. Printers' mistakes were frequent and the quality of printing in general was poor and since he had to prepay for the issues, acceptable or not, he was placed in a financial disadvantage. His most successful publishing effort was the *Mathematical Diary* which ran from 1825 to 1832. Perhaps the times were becoming more conducive to mathematical exposition. However, he retained the journal's editorship for only a year before relocating from New York City to Rutgers. His wife, Ann, refused to live in large cities, forcing him to maintain two households. Adrain was always pressed for funds to support his large family (seven children) and his work.

Two circumstances contributed to the failure to receive more acclaim for his work on error distributions. First, it is known that he was aware of the work of Legendre and Gauss who also concerned themselves with error distributions and MLS; however authorities who have examined this work carefully, support Adrain's originality and priority [**11**]. By modern standards of rigor, his derivation of the error distribution is flawed in its premises. In accepting Bowditch's initial assumption that the error in determining a straight line is proportional to its length, Adrain placed himself in the position of a nineteenth century surveyor when the determination of a straight line through forest and over rugged terrain required many sightings, each subject to error. The physical and technical difficulties of American frontier surveying have recently been commented upon in Linklater's *Measuring America* [**24**]. [The author encountered this task while chaining land for jungle settlements in Southeast Asia during the 1960s]. Thus, in a practical field situation, error could be considered proportional to length. Adrain also assumed that two sighting errors (length) were independent; however, in his derivation, he makes them proportional to each other—violating independence. Yes, under rigorous inspection his methods were faulty. But in this time of mathematical exploration and adventurism the methods of many mathematicians, including even those of Gauss and Legendre, were suspect. Often in the case of more well known mathematicians, their reputations deflected open criticism. Formalism and rigor were sacrificed for immediate, useable, results.

Adrain's place in history must be judged by the conditions and standards of his time. Julian Coolidge after closely examining the mathematical career of Robert Adrain and his colleagues concluded that:

> There can be no question as to his outranking every American mathematician who was really his contemporary [**11**, p. 75].

Working in isolation from the great mathematics research centers of Europe and with little institutional and societal support he still identified and probed some of the outstanding mathematical issues of the early nineteenth century. Just how great a mathematician he was is still open to judgment. However less debatable are his numerous contributions to the cause of promoting mathematics and popularizing its study in early American society. The real mystery is "Why isn't Robert Adrain better recognized for his work as a mathematics educator?" Ultimately, he should be recognized primarily as a mathematics educator, perhaps America's first mathematics educator.

REFERENCES

1. C. Abbee, A historical note on the Method of Least Squares, *American Journal of Science and Arts* **1** (1871) 411–415.
2. G. Bowditch Adrain, Robert Adrain, LL.D, *The United States Magazine and Democratic Review* (June, 1844) 646–656.
3. R. Adrain, Investigation of the figure of the earth, and the gravity in different latitudes, *Transactions of the American Philosophical Society* **1** (1818) 118–135.
4. D. C. Arney, J. Albree, V. F. Rickey, *A Station Favorable to the Pursuits of Science: Primary Materials in the History of Mathematics at the United States Military Academy*, American Mathematical Society (2000).
5. *Analyst, The*; or *Mathematical Museum*, Robert Adrain (ed). Philadelphia: William P. Farrand and Co., (1808–09) 813–814. See Karpenski pp. 581–583.
6. M. J. Babb, Robert Adrain—man and mathematician, *The General Magazine and Historical Chronicle* **28** (1926) 272–284.
7. F. Cajori, *The Teaching and History of Mathematics in the United States*, Washington, D.C.: Government Printing Office (1890).

8. S. L. Chapin, Geodesy, *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, I. Grattan-Guinness (ed.), Baltimore, MD: The John Hopkins Press (2003) 1089–1100.
9. P. C. Cohen, *A Calculating People: the Spread of Numeracy in Early America*, Chicago: The University of Chicago Press, 1982.
10. R. Cooke, *The History of Mathematics*, New York: John Wiley, 1997.
11. J. L. Coolidge, Robert Adrain and the beginnings of American mathematics, *American Mathematical Monthly* **33** (1926) 61–76.
12. J. Dutka, Robert Adrain and the Method of Least Squares, *Archive for History of Exact Science* **41** (1990) 171–184.
13. C. Elliott, Robert Adrain, *Biographical Dictionary of American Science: The Seventeenth Through Nineteenth Centuries*, Westport, CN. Greenwood Press (1979) 12–13.
14. J. V. Grabiner, Mathematics in America: the first hundred years, *The Bicentennial Tribute to American Mathematics 1776–1976*, Dalton Tarwater (ed.), Mathematical Association of America (1977) 9–25.
15. D. Hart, Historical sketch of American mathematical periodicals, *The Analyst* **2** (Sept. 1875) 131–138.
16. B. Hayes, Science on the farther shore, *American Scientist Online* **90** (2002) 6 pages. `http://www.americanscientist.org`.
17. J. F. W. Herschel, Review: Letters a' S.A.R. le regnant de Saxe Coboug et Gotha, *Edinburgh Review* **83** (1850) 1–57.
18. E. R. Hogan, George Baron and the *Mathematical Correspondent*, *Historia Mathematica* **3** (1976) 403–415.
19. E. R. Hogan, Robert Adrain: American mathematician, *Historia Mathematica* **4** (1977) 157–172.
20. A. G.Howson, *A History of Mathematics Education in England*, New York: Cambridge University Press (1982).
21. C. Hutton, *A Course of Mathematics in Two Volumes for the Use of Academics as Well as Private Tuition*, 2 vol., 3rd American ed., New York: Samuel Campbell and Son (1822).
22. P. Jones and A. Coxford, Mathematics in the evolving schools, *A History of Mathematics in the United States and Canada*, National Council of Teachers of Mathematics Yearbook **32** (1970) 11–86.
23. L. C. Karpinski, *Bibliography of Mathematical Works Printed in America Through 1850*, Ann Arbor: University of Michigan Press (1940).
24. A. Linklater, *Measuring America*, New York: Walker Publishing, (2002).
25. MacTutor History of Mathematics website. See `http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Adrain.html` [June 5, 2008].
26. *Mathematical Correspondent, The*, George Baron (ed. 1804–1807); Robert Adrain (ed. 1808). Vol. I printed in New York: Sage and Clough, vol. II printed in Reading, PA: Gottlob Jungman. See Karpinski, pp. 591–592.
27. *Mathematical Diary, The*, Robert Adrain (ed. 1825); James Ryan (ed. 1826–1832), 2 vols. New York: James Ryan. See Karpinski, pp. 591–592.
28. T. Perl, The Ladies Diary or the Woman's Almanack, 1704–1841, *Historia Mathematica* **6** (1979) 36–53.
29. G. Pettengill, Robert Adrain, Reading Mathematician, *Historical Review of Berks County* (July 1943) 111–114.
30. J. Ryan, *An Elementary Treatise on Algebra*, New York: Collins and Hannay (1824).
31. D. E. Smith, and J. Ginsburg, *A History of Mathematics in America before 1900*, Mathematical Association of America (1934).
32. G. Stanic and J. Kilpatrick (eds.), *A History of School Mathematics*, 2 vols. National Council of Teachers of Mathematics (2003).
33. S. Stigler (ed.), *American Contributions to Mathematical Statistics in the Nineteenth Century*, 2 vols. New York: Arno Press (1980).
34. S. Stigler, *The History of Statistics: The Measurement of Uncertainty Before 1900*, Cambridge, MA: Belknap Press of Harvard University (1986).
35. D. E. Struik, Robert Adrain, Dictionary of Scientific Biography, vol. I, Charles Gillepsie (ed.). New York: Charles Scribner's Sons (1970) 65–66.
36. D. E. Struik, *Yankee Science in the Making*, New York: Dover (1991).
37. J. Suzuki, *A History of Mathematics*, Upper Saddle River, NJ: Prentice Hall (2002) 588–594.
38. F. Swetz, Back to the present: ruminations on an old arithmetic text, *Mathematics Teacher* **86** (Sept. 1993) 491–494.
39. D. Zitarelli, The Bicentennial of American mathematics journals, *College Mathematics Journal* **36** (2005) 2–15.

Touching the $\mathbb{Z}_2$ in Three-Dimensional Rotations

VESNA STOJANOSKA
Northwestern University
Evanston IL 60201
vesna@math.northwestern.edu

ORLIN STOYTCHEV
American University in Bulgaria
2700 Blagoevgrad, Bulgaria
and
Institute for Nuclear Research
1784 Sofia, Bulgaria
ostoytchev@aubg.bg

Rotations, belts, braids, spin-1/2 particles, and all that

The space of all three-dimensional rotations is usually denoted by $SO(3)$. This space has a well-known and fascinating topological property—a complete rotation of an object is a motion which may or may not be continuously deformable to the trivial motion (i.e., no motion at all) but the composition of two motions that are not deformable to the trivial one gives a motion, which is. (Here and further down by "complete rotation" we will mean taking the object at time $t = 0$ and rotating it as t changes from 0 to 1 arbitrarily around a fixed point, so that at $t = 1$ the object is brought back to its initial orientation.) A rotation around some fixed axis by 360° cannot be continuously deformed to the trivial motion, but it can be deformed to a rotation by 360° around any other axis (in any direction). However, a rotation by 720° is deformable to the trivial one.

You may try to see some of this at home by performing a complete rotation of a box, keeping one of the vertices fixed. Let us first rotate the box around one of the edges and then try to deform this motion to the trivial one. If you follow a vertex on one of the non-fixed edges, it will trace a large circle on a sphere. Now, for any complete rotation of the box (around the same fixed vertex) the vertex we are following will have to trace some closed path on that sphere. So as you try to deform continuously the initial motion to the trivial one, the vertex you are tracking will have to trace smaller and smaller paths, starting from the large circle and ending with the constant path, which is just the initial and final point. As you do this, one of the other vertices, which was left fixed by the initial motion, will start tracing larger and larger paths approaching a large circle on a sphere. Thus in effect, trying to contract a rotation around one of the edges to the trivial one, you only managed to deform it to a rotation around a different edge. There is some intrinsic "topological obstacle" to contracting such motions. You would need a considerable imagination to see the second property—if your initial motion consists of two full rotations around some axis, it can be deformed to the trivial motion. There are a few famous "tricks" relying on this property, most notably "Dirac's belt trick" and "Feynman's plate trick." In the "belt trick" you fasten one end of the belt and rotate the other end (the buckle) by 720°. Then, without changing the orientation of the buckle, you untwist the belt, by passing it around the buckle. (See a nice animation on Greg Egan's web-page [**6**] and Java applets analyzing the "tricks" by Bob Palais [**9**].) The "plate trick" is essentially the same. You put a (full) plate onto your palm and, without moving your feet, rotate it by 720°, at the same time moving it under your armpit and then over your head. You will end up in your initial position, your arm and body untwisted.

These experiments should leave you with a few questions: Is the complete rotation around one axis really not contractible to the trivial motion? If you have two arbitrary motions that are not contractible, can you always deform one to the other? If you compose two of the latter do you always get a motion that is contractible? (The affirmative answer to the last question actually will follow from the affirmative answer to the previous one together with the "belt trick" effect.) We will describe an experiment, which could be called the "braid trick" and which will give us enough machinery to answer these questions rigorously. In the process, we exhibit an intriguing relation between three-dimensional rotations and braid groups.

Complete rotations of an object are in one-to-one correspondence with closed paths in $SO(3)$. Two closed paths in a topological space with the same initial and final point (base point) are called *homotopic* if one can be continuously deformed to the other. Since homotopy of paths is an equivalence relation, all paths fall into disjoint equivalence classes. The set of homotopy classes of closed paths becomes a group when one takes composition of paths as the multiplication and tracing a path in the opposite direction as the inverse. This group, noncommutative in general, is one of the most important topological invariants of a space and was first introduced by Poincaré. It is called the *fundamental group* or the *first homotopy group* and is denoted by π_1. Thus for the space of three-dimensional rotations the topological property discussed so far is written in short as $\pi_1(SO(3)) \cong \mathbb{Z}_2$. This means that all closed paths in $SO(3)$ starting and ending at the same point, e.g., the identity, fall into two homotopy classes—those that are homotopic to the constant path and those that are not. Composing two paths from the second class yields a path from the first class.

A topological space with a fundamental group $\mathbb{Z}_2$ is a challenge to the imagination—it is easy to visualize spaces with fundamental group $\mathbb{Z}$ (the punctured plane), or $\mathbb{Z} \star \mathbb{Z} \cdots \star \mathbb{Z}$ (plane with several punctures), or even $\mathbb{Z} \oplus \mathbb{Z}$ (torus), but there is no subspace of $\mathbb{R}^3$ whose fundamental group is $\mathbb{Z}_2$.

The peculiar structure of $SO(3)$ plays a fundamental role in our physical world. There are exactly two principally different types of elementary particles, bosons, having integer spin, and fermions, having half-integer spin, with very distinct physical properties. The difference can be traced to the fact that the quantum state of a boson is described by a (possibly multi-component) wave function, which remains unchanged when a full (360°) rotation of the coordinate system is performed, while the wave function of a fermion gets multiplied by -1 under a complete rotation. Somewhat loosely speaking, the second possibility comes from the fact that only the modulus of the wave function has a direct physical meaning. Mathematical physicists have realized long ago **[11, 2]** that the wave function has to transform properly only under the action of transformations that are in a small neighborhood of the identity. When a "large" transformation is performed on the wave function, like a rotation by 360°, it can be done by a sequence of "small" transformations, but the end point—the transformed wave function—need not coincide with the initial one. On the other hand, if you take a closed path in $SO(3)$ which remains in a small neighborhood of the identity, the transformed wave function at the end must coincide with the initial one. In fact what is important is whether the closed path is contractible to the identity or not. It is quite obvious from continuity considerations that the end-point wave function must coincide with the initial one if the path in $SO(3)$ is contractible. Thus when you do two full rotations, i.e., rotation by 720°, the wave function should come back to the initial one which implies that the transformation, corresponding to a 360°–rotation must be of order 2.

There are several standard ways of showing that $\pi_1(SO(3)) \cong \mathbb{Z}_2$. The one that is best known uses substantially Lie group and Lie algebra theory. The space $SO(3)$ can be thought of as the space of 3×3 real orthogonal matrices with determinant 1. It has the structure of a closed three-dimensional smooth manifold embedded in $\mathbb{R}^9$ (a higher-dimensional analog of a closed smooth surface embedded in $\mathbb{R}^3$). It is also a group and the group operations are smooth maps. Such spaces are called Lie groups. Another Lie group, very closely related to $SO(3)$ is $SU(2)$—the group of 2×2 complex unitary matrices with determinant 1. It is relatively easy to see that topologically $SU(2)$ is the three-dimensional sphere S^3. Locally the two groups are identical, i.e., one can find a bijection between open neighborhoods of the identities of both, which is a group isomorphism and a (topological) homeomorphism. Globally, however, this map extends to a 2–1 homomorphism $SU(2) \to SO(3)$, sending any two antipodal points on $SU(2)$ to a single point on $SO(3)$. In topological terms this map is called a double covering of $SO(3)$. The topology of $SO(3)$ can now be easily understood—it is the three-dimensional sphere S^3 with antipodal points identified.

In the present paper we describe an alternative way of "seeing" and proving that $\pi_1(SO(3)) \cong \mathbb{Z}_2$. It does not use Lie groups or even matrices. It is purely algebraic-topological in nature and very visual. It displays a simple connection between full rotations (closed paths in $SO(3)$) and braids. We believe that this is an interesting way of demonstrating a nontrivial topological result to students in introductory geometry and topology courses as well as a suitable way of sparking interest in braids and braid groups, which appear naturally in various mathematical problems, from algebraic topology through operator algebras to robotics and cryptography.

Relationships between braids and homotopy groups appear at different levels. To begin with, braid groups can be defined as the fundamental groups of certain configuration spaces. Braids have been applied (see, e.g., **[4]**) to determining homotopy groups of the sphere S^2. In this paper, we present yet another, simple connection between braid groups and a fundamental group.

The goal of this paper is mostly pedagogical—presenting in a self-contained and accessible way a set of results that are basically known to algebraic topologists and people studying braid groups. The fact that the first homotopy group of $SO(3)$ can be related to spherical braids is a special case (in disguise) of the following general statement **[7]**: "The configuration space of three points on an r-sphere is homotopically equivalent to the Stiefel manifold of orthogonal two-frames in $r + 1$-dimensional Euclidean space." Fadell **[7]** considers a particular element of $\pi_1(SO(3))$ and uses the fact that it has order 2 to prove a similar statement for a corresponding braid. Our direction is the opposite—we analyze braids to deduce topological properties of $SO(3)$.

In the next section we describe a simple experiment that actually demonstrates the $\mathbb{Z}_2$ in three-dimensional rotations. Then in section 3 we give a formal treatment of that experiment. We construct a map from $\pi_1(SO(3))$ into a certain factorgroup of a subgroup of the braid group with three strands. We prove that this map is an isomorphism and that the image is $\mathbb{Z}_2$.

The braid trick

Take a ball (a tennis ball will do) and attach three strands to three different points on its surface. Attach the other ends of the strands to three different points on the surface of your desk (FIGURE 1). Perform an arbitrary number of full rotations of the ball around arbitrary axes. You will get a plaited "braid". (When you do the rotations, your strands will have to be loose enough. Still, if you are performing just rotations of the ball without translational motions, what you will get is a "braid" and not the more complicated object "tangle" in which each strand can be knotted by itself. Even though this more complicated situation can be handled easily, we prefer to avoid it.) Now keep the orientation of the ball fixed. If the total number of full rotations is even, you can always unplait the "braid" by flipping strands around the ball. If the number of rotations is odd you will never be able to unplait it, but you can always reduce it to one simple configuration, e.g., the one obtained by rotating the ball around the first point and twisting the second and third strands around each other.

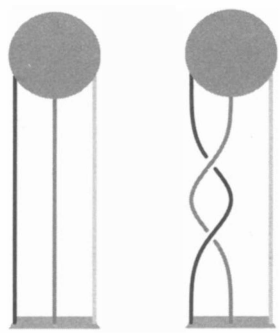


Figure 1 Rotating a ball with strands attached.

As we might expect, rotations that can be continuously deformed to the trivial rotation (i.e., no rotation) lead to trivial braiding. At this point we can only conjecture from our experiment that the fundamental group of $SO(3)$ contains $\mathbb{Z}_2$ as a factor.

Relating three-dimensional rotations to braids

With each closed path in $SO(3)$ we associate three closed paths in $\mathbb{R}^3$ starting at the sphere with radius 1 and ending at the sphere with radius $1/2$. We may think of continuously rotating a sphere from time $t = 0$ to time $t = 1$ so that the sphere ends up with the same orientation as the initial one. Simultaneously we shrink the radius of the sphere from 1 to $1/2$ (see FIGURE 2). Any three points on the sphere will trace three continuous paths in $\mathbb{R}^3$, which do not intersect each other. Furthermore, for fixed t the three points on these paths lie on the sphere with radius $1 - t/2$. To formalize things, let $\omega(t)$, $t \in [0, 1]$ be any continuous path in $SO(3)$ with $\omega(0) = \omega(1) = I$. $\omega(t)$ acts on vectors (points) in $\mathbb{R}^3$. Take three initial points in $\mathbb{R}^3$, e.g., $\mathbf{x}_0^1 = (1, 0, 0)$, $\mathbf{x}_0^2 = (-1/2, \sqrt{3}/2, 0)$, $\mathbf{x}_0^3 = (-1/2, -\sqrt{3}/2, 0)$. Define three continuous paths by

$$\mathbf{x}^i(t) := (1 - t/2)\omega(t)(\mathbf{x}_0^i), \quad t \in [0, 1], \quad i = 1, 2, 3.$$

In this way we get an object that will be called a *spherical braid*—several distinct points on a sphere and the same number of points, in the same positions, on a smaller sphere, connected by strands in such a way that the radial coordinate of each strand is monotonic in t.

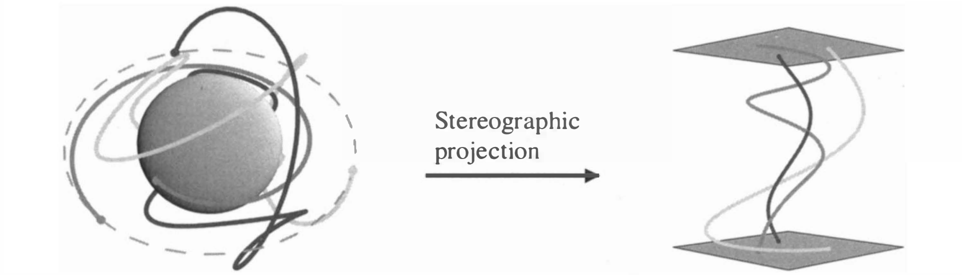


Figure 2 A "spherical braid" and a normal braid.

We can map our spherical braid to a conventional one using stereographic projection (FIGURE 2). First we choose a ray starting at the origin and not intersecting any strand. The ray intersects each sphere at a point, which we can consider as the "north pole". Then we map stereographically, with respect to its "north pole," each sphere with radius $1/2 \leq \rho \leq 1$ (minus its "north pole") to a corresponding (horizontal) plane. Finally we define the z-coordinate of the image to be $z = -\rho$.

Recall the usual notion of ***braids***, introduced by Artin **[1]**. (See also **[4]** for a contemporary review of the theory of braids and its relations to other subjects.) We take two planes in $\mathbb{R}^3$, let's say parallel to the XY plane, fix n distinct points on each plane and connect each point on the lower plane with a point on the upper plane by a continuous path (strand). The strands do not intersect each other. In addition the z-coordinate of each strand is a monotonic function of the parameter of the strand and thus z can be used as a common parameter for all strands. Two different braids are considered equivalent or *isotopic* if there exists a homotopy of the strands (keeping the endpoints fixed), so that for each value of the homotopy parameter s we get a braid, for $s = 0$ we get the initial braid and for $s = 1$ the final one. When the points on the lower and the upper plane have the same positions (their x and y coordinates are the same), we can multiply braids by stacking one on top of the other. Considering classes of isotopic braids with the multiplication just defined, the ***braid group*** is obtained. Artin showed that the braid group B_n on n strands has a presentation with $n - 1$ generators and a simple set of relations—Artin's braid relations. We give them for the case $n = 3$ since this is the one we are mostly interested in. In this case the braid group B_3 is generated by the generators σ_1, corresponding to twisting of the first and the second strands, and σ_2, corresponding to twisting of the second and the third strands (the one to the left always passing behind the one to the right) (FIGURE 3). These generators are subject to a single braid relation (FIGURE 4):

$$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1 \tag{1}$$

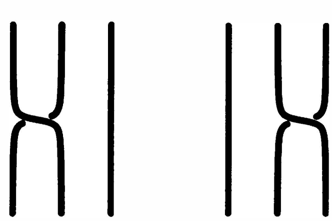


Figure 3 The generators σ_1 and σ_2 of B_3.

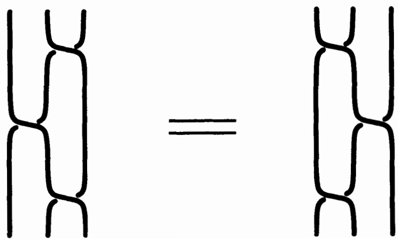


Figure 4 The braid relation for B_3.

We say that B_3 has a *presentation* with generators σ_1 and σ_2 and defining relation given by Equation 1, or in short:

$$B_3 = \langle \sigma_1, \sigma_2; \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \rangle \tag{2}$$

In our case, since a full rotation of the sphere returns the three points to their original positions, we always get *pure braids*, i.e., braids for which any strand connects a point on the lower plane with its translate on the upper plane. Pure braids form a subgroup of B_3 which is denoted by P_3. Note that intuitively there is a homomorphism π from B_3 to the symmetric group S_3 since any braid from B_3 permutes the three points. Formally we define π on the generators by

$$\pi(\sigma_1)(1, 2, 3) = (2, 1, 3), \quad \pi(\sigma_2)(1, 2, 3) = (1, 3, 2) \tag{3}$$

and then extend it to the whole group B_3 (it is important that π maps Equation 1 to the trivial identity). Pure braids are precisely those that do not permute the points and therefore we can give the following algebraic characterization of P_3:

$$P_3 := \operatorname{Ker} \pi.$$

Alternatively, S_3 is the quotient of B_3 by the additional equivalence relations $\sigma_i^2 \sim I$, $i = 1, 2$ and if N is the minimal normal subgroup containing σ_i^2, then $\pi : B_3 \to B_3/N$ is the natural projection. It is then easy to see that the kernel of π has to be a product of words of the following type:

$$\sigma_{i_1}^{\pm 1}\sigma_{i_2}^{\pm 1} \cdots \sigma_{i_k}^{\pm 1}\sigma_{i_{k+1}}^{\pm 2}\sigma_{i_k}^{\pm 1} \cdots \sigma_{i_2}^{\pm 1}\sigma_{i_1}^{\pm 1}.$$

The whole subgroup P_3 can in fact be generated by the following three *twists* (FIGURE 5)

$$a_{12} := \sigma_1^2, \quad a_{13} := \sigma_2\sigma_1^2\sigma_2^{-1} = \sigma_1^{-1}\sigma_2^2\sigma_1, \quad a_{23} := \sigma_2^2. \tag{4}$$

In our construction so far we mapped any closed path in $SO(3)$ to a spherical braid and then, using stereographic projection, to a conventional pure braid. The last map, however, depends on a choice of a ray in $\mathbb{R}^3$ and, what is worse, spherical braids that are isotopic (in the obvious sense) may map to nonisotopic braids. To mend this, we will identify certain classes of braids in P_3. Namely, we introduce the following equivalence relations (see FIGURE 6):

$$r_1 := \sigma_1\sigma_2^2\sigma_1 \sim I, \quad r_2 := \sigma_1^2\sigma_2^2 \sim I, \quad r_3 := \sigma_2\sigma_1^2\sigma_2 \sim I. \tag{5}$$

In our model with the tennis ball the elements r_i, $i = 1, 2, 3$ correspond to *flips* of the ith strand above and around the ball. Such motions lead to isotopic spherical braids, as will be shown later. (The choice of these particular three flips given in Equation 5 is

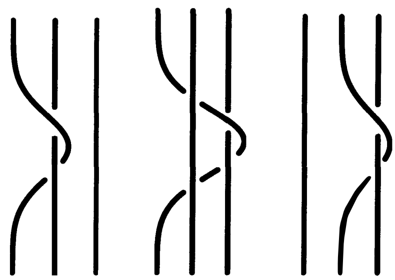


Figure 5 The generators a_{12}, a_{13}, and a_{23} of P_3.

based on the following intuition, coming from the experiment—thinking of the three strands of the trivial braid as arranged in a circle, we pull one of them out and flip it above and around the ball clockwise to obtain one of the r_i or counterclockwise to obtain its inverse. Thus in FIGURE 6 the middle strand is in the background, while the first and third are in the foreground. We do not take "more complicated" elements, like e.g., $\sigma_2^2\sigma_1^2$ which would correspond to first pulling the middle strand between the other two to the foreground and then performing the flip r_1, i.e., $\sigma_2^2\sigma_1^2$ is obtained from r_1 by conjugating it with σ_1 and its inverse.)

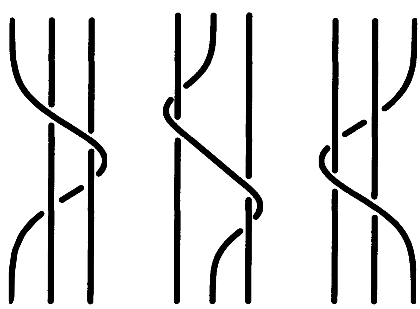


Figure 6 The flips r_1, r_2, and r_3.

Note. When any strand in any part of the spherical braid crosses the ray which we use for the stereographic projection, that projection will map the spherical braid to a different (Artin) braid, which we should consider as identical with the initial one. This means that we have to factorize by the normal closure in B_3 (not in P_3!) of the generators r_i, $i = 1, 2, 3$, i.e., the smallest normal subgroup in B_3 containing these three generators. This would then allow us to set to I any r_i (or its inverse) in any part of a word. We see easily that only one of the generators is needed then, since the other two will be contained in the normal closure of the first. We noticed experimentally, however, that we managed to untie any trivial braid just by a sequence of the three flips r_i defined in Equation 5 and their inverses, performed at the end of the braid. At the same time a nontrivial braid, corresponding to an odd number of rotations, cannot be untied even if we allow flips in any part of the braid. This can only be true if the flips r_i generate a normal subgroup in B_3 (which of course then coincides with the normal closure of any of the r_i and is also normal in P_3).

LEMMA 1. *The subgroup $R \subset P_3$, generated by r_1, r_2, r_3 is normal in B_3.*

Proof. We need to show that we can represent all conjugates of r_i with respect to the generators of B_3 and their inverses as products of the r_i and their inverses.

Straightforward calculations, using repeatedly Artin's braid relation (Equation 1) give the following identities:

$$\begin{aligned}
\sigma_1 r_1 \sigma_1^{-1} &= r_2, & \sigma_2 r_1 \sigma_2^{-1} &= \sigma_2^{-1} r_1 \sigma_2 = r_1,\\
\sigma_1 r_2 \sigma_1^{-1} &= r_2 r_1 r_2^{-1}, & \sigma_2 r_2 \sigma_2^{-1} &= r_3,\\
\sigma_1 r_3 \sigma_1^{-1} &= \sigma_1^{-1} r_3 \sigma_1 = r_3, & \sigma_2 r_3 \sigma_2^{-1} &= r_1^{-1} r_2 r_1 = r_3 r_2 r_3^{-1},\\
\sigma_1^{-1} r_1 \sigma_1 &= r_1^{-1} r_2 r_1, & \sigma_1^{-1} r_2 \sigma_1 &= r_1,\\
\sigma_2^{-1} r_2 \sigma_2 &= r_1 r_3 r_1^{-1} = r_2^{-1} r_3 r_2, & \sigma_2^{-1} r_3 \sigma_2 &= r_2.
\end{aligned} \tag{6}$$

We demonstrate as an example the proof of the first identity in the second line. We have

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1 &= \sigma_2\sigma_1\sigma_2\\
\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1\\
\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1\\
\sigma_1\sigma_2^2\sigma_1\sigma_2^2 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1\\
\sigma_1\sigma_2^2\sigma_1 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1\sigma_2^{-2}\\
\sigma_1^3\sigma_2^2\sigma_1^{-1} &= \sigma_1^2\sigma_2^2\sigma_1\sigma_2^2\sigma_1\sigma_2^{-2}\sigma_1^{-2},
\end{aligned}$$

and therefore

$$\sigma_1 r_2 \sigma_1^{-1} = \sigma_1 \cdot \sigma_1^2\sigma_2^2 \cdot \sigma_1^{-1} = \sigma_1^2\sigma_2^2 \cdot \sigma_1\sigma_2^2\sigma_1 \cdot \sigma_2^{-2}\sigma_1^{-2} = r_2 r_1 r_2^{-1}. \qquad \blacksquare$$

By suitable full rotations we obtain all generators of P_3. For example, a_{12} is obtained by rotating around the vector $\mathbf{x}_0^3 = (-1/2, -\sqrt{3}/2, 0)$ and it twists the first and the second strand. Furthermore, homotopies between closed paths in $SO(3)$ correspond to isotopies of the spherical braids and thus homotopic closed paths in $SO(3)$ will be mapped to the same element in the factorgroup P_3/R. Hence we have a surjection $\pi_1(SO(3)) \to P_3/R$.

PROPOSITION 1. *The factorgroup P_3/R is isomorphic to $\mathbb{Z}_2$.*

Proof. To make notation simpler we use the same letter to denote both a representative of a class in P_3/R and the class itself, hoping that the meaning is clear from the context. In P_3/R we have

$$\sigma_1\sigma_2^2 = \sigma_1^{-1} = \sigma_2^2\sigma_1,$$

and

$$\sigma_2\sigma_1^2 = \sigma_2^{-1} = \sigma_1^2\sigma_2.$$

The following sequence of identities follow one from another:

$$\begin{aligned}
\sigma_2\sigma_1^2 &= \sigma_1^2\sigma_2, & \sigma_1\sigma_2\sigma_1^2 &= \sigma_1^3\sigma_2,\\
\sigma_2\sigma_1\sigma_2\sigma_1 &= \sigma_1^3\sigma_2, & \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 &= \sigma_1^4\sigma_2,\\
\sigma_1\sigma_2^2\sigma_1\sigma_2 &= \sigma_1^4\sigma_2, & I &= \sigma_1^4.
\end{aligned}$$

We have used twice the braid relation (Equation 1) and the first equivalence relation in Equation 5. In a completely analogous way we prove

$$\sigma_2^4 = I.$$

Combining the last two results with the equivalence relations (Equation 5) we finally get

$$\sigma_1^2 = \sigma_1^{-2} = \sigma_2^2 = \sigma_2^{-2}. \tag{7}$$

It is now clear that in P_3/R the three generators, defined in Equation 4 reduce to one element of order 2. Therefore they generate $\mathbb{Z}_2$. This completes the proof. ■

So far we have constructed a map $\pi_1(SO(3)) \to P_3/R$, which is onto by construction, and we have shown that the image is isomorphic to $\mathbb{Z}_2$. To show that this map is actually an isomorphism, we only need:

PROPOSITION 2. *The map $\pi_1(SO(3)) \to P_3/R$ is a monomorphism.*

Proof. It suffices to show that if a closed continuous path in $SO(3)$ is mapped to a braid in R, then this path is homotopic to the constant path. The proof basically reduces to the following observation — any spherical braid which is pure (the strands connect each point on the outer sphere with the same point on the inner sphere) determines a closed path in $SO(3)$. Two isotopic spherical pure braids determine homotopic closed paths in $SO(3)$. Indeed, recall that for a spherical braid we can parametrize the points on each strand with a single parameter t and that for a fixed t all three points lie on a sphere with radius $1 - t/2$. These three ordered points $\mathbf{x}^i(t)$, $i = 1, 2, 3$ give for every fixed t a nondegenerate triangle, oriented somehow in $\mathbb{R}^3$. Let $\mathbf{l}(t)$ be the vector, connecting the center of mass of the triangle with the vertex $\mathbf{x}^1(t)$, i.e., $\mathbf{l}(t) = \mathbf{x}^1 - (\mathbf{x}^1(t) + \mathbf{x}^2(t) + \mathbf{x}^3(t))/3$ and define $\mathbf{e}^1(t) := \mathbf{l}(t)/||\mathbf{l}(t)||$. Let $\mathbf{e}^3(t)$ be the unit vector, perpendicular to the plane of the triangle, in a positive direction relative to the orientation $(1, 2, 3)$ of the boundary. Finally, let $\mathbf{e}^2(t)$ be the unit vector, perpendicular to both $\mathbf{e}^1(t)$ and $\mathbf{e}^3(t)$, so that the three form a right-handed frame. Then there is a unique element $\omega(t) \in SO(3)$ sending the vectors $\mathbf{e}_0^1 = (1, 0, 0)$, $\mathbf{e}_0^2 = (0, 1, 0)$, $\mathbf{e}_0^3 = (0, 0, 1)$ to the triple $\mathbf{e}^i(t)$. According to our definitions, $\omega(0) = \omega(1) = I$ and we get a continuous function $\omega : [0, 1] \to SO(3)$, where continuity should be understood relative to some natural topology on $SO(3)$, e.g., the strong operator topology.

Recall that for any spherical braid the ith strand ($i = 1, 2, 3$) starts at the point $\mathbf{x}_0^i$ and ends at the point $\mathbf{x}_0^i/2$. If we have two isotopic spherical braids, by definition there are continuous functions $\mathbf{x}^i(t, s)$, $i = 1, 2, 3$, such that $\mathbf{x}^i(t, s)$ is a braid for any fixed $s \in [0, 1]$, $\mathbf{x}^i(0, s) = \mathbf{x}_0^i$, $\mathbf{x}^i(1, s) = \mathbf{x}_0^i/2$, $\mathbf{x}^i(t, 0)$ give the initial braid and $\mathbf{x}^i(t, 1)$ give the final braid. By assigning an element $\omega(t, s)$ to any triple $\mathbf{x}^i(t, s)$ as described, we get a homotopy between two closed paths in $SO(3)$.

Let $\omega'(t)$ be a closed path in $SO(3)$ which is mapped to a braid b in the class $r_1 \in R$. We can construct a spherical braid, whose image is isotopic to that braid. Let $\mathbf{z}$ be the point on the unit sphere with respect to which we perform the stereographic projection. This can always be chosen to be the north pole or a point very close to the north pole (in case a strand is actually crossing the axis passing through the north pole). Note that the points $\mathbf{x}_0^i$, $i = 1, 2, 3$ are on the equator. Construct a simple closed path on the unit sphere starting and ending at $\mathbf{x}_0^1$ and going around $\mathbf{z}$ in a negative direction (without crossing the equator except at the endpoints). Thus we have two continuous functions $\varphi(t)$, $\theta(t)$, $t \in [0, 1]$—the spherical (angular) coordinates describing this path. Let $\mathbf{x}^1(t)$ be the point in $\mathbb{R}^3$ whose spherical coordinates are $\rho(t) := 1 - t/2$, $\varphi(t)$, $\theta(t)$ and let $\mathbf{x}^i(t) := (1 - t/2)\mathbf{x}_0^i$, $i = 2, 3$. These three paths give the required spherical braid. It is isotopic to the trivial braid, coming from the constant path in $SO(3)$, and at the same time it is isotopic to the preimage of b under the stereographic projection. In this way we see that $\omega'(t)$ must be homotopic to the constant path. Obviously a similar argument holds with r_1 replaced by r_2 and r_3 or the inverses. Since any element in R

is a product of these generators, and since products of isotopic braids give isotopic braids, this completes the proof. ■

Further discussion, results, and generalizations

When we look at a complicated braid that has been plaited by numerous different rotations of our ball, it may seem difficult to tell whether it can be untied (by performing flips r_i) or not. Actually, there is a simple criterion to determine this. Assume that the braid is represented as some word in the Artin generators:

$$b = \sigma_1^{m_1}\sigma_2^{n_1}\sigma_1^{m_2}\sigma_2^{n_2}\ldots\sigma_1^{m_k}\sigma_2^{n_k}. \tag{8}$$

Define the following invariant, called the *length* of the braid:

$$p(b) := m_1 + n_1 + m_2 + n_2 + \cdots + m_k + n_k. \tag{9}$$

Note that m_i and n_i can be any integers (positive, negative or zero). We observe that the number $p(b)$ is invariant for Artin's braid, since applying the braid relation (Equation 1) inside any word does not change $p(b)$ of that word. Next, since we know that our braid is pure, it can be written as a product of the generators a_{12}, a_{13}, and a_{23} defined in Equation 4 and their inverses. Note that each of these generators has $p(b) = 2$. So we conclude that $p(b)$ is even. Now, if $p(b) = 0$ (mod 4) this means that b is a product of even number of the generators a_{ij} (and their inverses). We saw in the proof of Proposition 1 that in P_3/R the three generators a_{ij} reduce to one element of order 2, so $p(b) = 0$ (mod 4) implies that b is trivial in P_3/R or can be untied by performing flips. On the other hand, if $p(b) = 2$ (mod 4), then b is a product of odd number of generators a_{ij} (and their inverses) and thus reduces to the single nontrivial element of P_3/R. In this way we have provided a (simple) algorithm solving the so-called *word problem* for P_3/R, i.e., one can decide in a finite number of steps algorithmically whether two words represent the same group element or not.

There is a more intriguing aspect of our "puzzle"—given a complicated braid which is trivial in P_3/R, can we provide a recipe for a sequence of flips r_i that will untie it? (When one experiments with the tennis ball one usually intuitively finds a sequence of flips, but can we program a computer to do it?) Mathematically the problem reduces to the following: given an element $b \in R \subset B_3$, which is written in terms of the generators of B_3, can we give an algorithm to rewrite this element in terms of the generators of R? The authors don't know the answer to this question, though it may be simple. We should point out that such questions about the braid group, its subgroups and factor-groups have sparked considerable interest, especially in connection with their possible use in cryptography (see, e.g., [**5**] for examples).

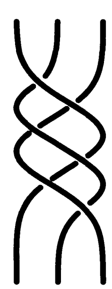


Figure 7 The full twist d in the case $n = 3$.

We can easily understand the "belt trick" or the "plate trick" using algebra. In our experiment with the ball let's perform two full rotations (full twists) around a vertical axis (FIGURE 8). A single full twist, as in FIGURE 7 leads to the braid $d := (\sigma_1\sigma_2)^3$. For two full twists, using twice Artin's braid relation, we get:

$$d^2 = (\sigma_1\sigma_2)^6 = (\sigma_2\sigma_1)^6 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1(\sigma_2\sigma_1)^3 = \sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1(\sigma_2\sigma_1)^3$$
$$= r_3\sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 = r_3\sigma_1^2\sigma_2\sigma_2\sigma_1\sigma_2^2\sigma_1 = r_3r_2r_1$$

Therefore we can unplait the braid d^2 by applying the sequence of flips $r_3^{-1}, r_2^{-1}, r_1^{-1}$ (in that order). Intuitively this is the same as flipping the whole bunch of three strands together above and around the ball. It is also obvious that it should not matter with which strand we start, so cyclic permutations of the above sequence of flips should also unplait the braid. If we look at some of the identities in Equations 6 we see indeed that $r_3r_2r_1 = r_2r_1r_3 = r_1r_3r_2$.

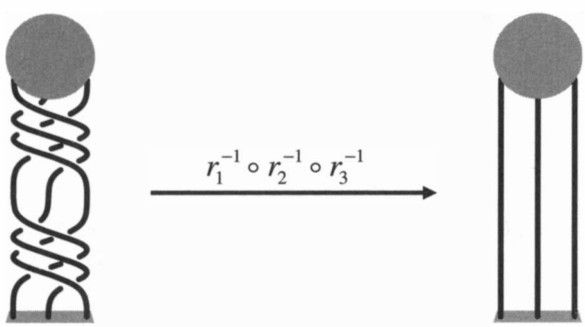


Figure 8 The "belt trick."

There is an obvious generalization of some of the results of the previous sections to the case $n > 3$. The minimal number of strands that is needed to capture the nontrivial fundamental group of $SO(3)$ is $n = 3$. When $n > 3$ any full rotation will give rise to a pure spherical braid but the whole group of pure braids will not be generated in this way. It is relatively easy to see that in this way, after projecting stereographically, we will obtain a subgroup of P_n, generated by a single *full twist* d of all strands around an external point and a set of n flips r_i:

$$d := (\sigma_1\sigma_2\cdots\sigma_{n-1})^n,$$
$$r_1 := \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1,$$
$$r_2 := \sigma_1^2\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2,$$
$$r_i := \sigma_{i-1}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_i, \quad i = 2, 3, \ldots n-1,$$
$$r_n := \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}.$$

FIGURE 7 shows a full twist for the case with 3 strands while FIGURE 9 shows a generic flip. Straightforward calculations give the following generalization of Lemma 1:

LEMMA 1′. *The subgroup $R \subset P_n$, generated by r_i, $i = 1, \ldots n$, is normal in B_n.*

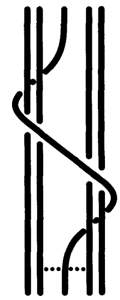


Figure 9 The flip r_i.

Proof. As in the proof of Lemma 1 we exhibit explicit formulas for the conjugates of all flips r_i:

$$\begin{aligned}
\sigma_j r_i \sigma_j^{-1} &= \sigma_j^{-1} r_i \sigma_j = r_i, \quad i - j > 1 \text{ or } j - i > 0,\\
\sigma_{i-1} r_i \sigma_{i-1}^{-1} &= r_i r_{i-1} r_i^{-1},\\
\sigma_{i-1}^{-1} r_i \sigma_{i-1} &= r_{i-1},\\
\sigma_i r_i \sigma_i^{-1} &= r_{i+1}, \quad i \leq n-1\\
\sigma_i^{-1} r_i \sigma_i &= r_i^{-1} r_{i+1} r_i, \quad i \leq n-1.
\end{aligned}$$

■

Let us denote by S the subgroup, generated by d and r_i. Using purely topological information, namely that $\pi_1(SO(3)) \cong \mathbb{Z}_2$, we can deduce the following generalization of Proposition 1:

PROPOSITION 1′. *The factorgroup S/R is isomorphic to $\mathbb{Z}_2$.*

An equivalent statement is that $d^2 \in R$.

Given a braid with more than 3 strands it is generally not simple to determine whether or not it belongs to the group S, or in other words whether or not it can be plaited when its strands are tied together at each end, starting from the trivial braid and performing flips r_i and twists d and their inverses (to the upper end). It turns out that this question is of importance for the construction of knitting machines and has been solved explicitly in [**10**]. The braid in FIGURE 10 for example can be obtained by a sequence of flips. Since the strands in this case stay in pairs we can think of them as representing *ribbons*. You can play around with this example by taking a paper strip, cutting two slits parallel to the long sides and trying to plait the shown configuration or you can look at Bar-Natan's gallery of knotted objects [**3**] from which the example was borrowed. In fact the "braided theta" in FIGURE 10 can be obtained by perform-

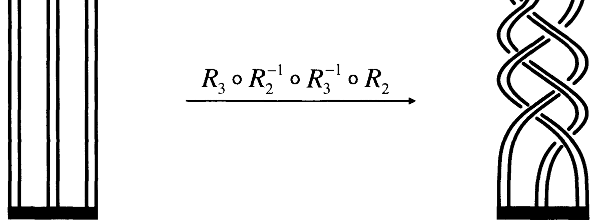


Figure 10 "Braided theta."

ing a sequence of *ribbon flips* R_1, R_2, R_3 and their inverses, which are similar to the ones in FIGURE 6 but performed on the 3 ribbons. By definition we have $R_i := r_{2i}r_{2i-1}$ and the effect of a flip R_i is similar to that of the usual flip r_i except that it twists the ith ribbon by 720° (counterclockwise). It is easier to find experimentally, rather than doing the algebra, that the "braided theta" in FIGURE 10 is the product $R_3R_2^{-1}R_3^{-1}R_2$.

If one tries to generalize the main result of this paper to higher dimensions, one would notice immediately that the isomorphism fails. On the one hand braids in higher than three-dimensional space can always be untangled. On the other hand the fundamental groups of $SO(n)$ are nontrivial. The reason for this failure is that we are able to attribute a path in $SO(3)$ to any spherical braid with 3 strands but this is not the case for $n > 3$ (4 points on S^3 may not determine an orientation of the orthonormal frame in $\mathbb{R}^4$.)

REFERENCES

1. E. Artin, Theory of braids, *Math. Ann.* **48**(1) (1947) 101–126.
2. V. Bargmann, On unitary ray representations of continuous groups, *Math. Ann.* **59**(1) (1954) 1–46.
3. `http://www.math.toronto.edu/~drorbn/Gallery/KnottedObjects/BraidedThetas/`
4. J. Birman and T. Brendle, Braids: a survey, E-print math.GT/0409205, `http://arxiv.org/abs/math/0409205`.
5. P. Dehornoy, Braid-based cryptography, *Contemp. Math.* **360** (2004) 5–33.
6. `http://gregegan.customer.netspace.net.au/APPLETS/21/21.html`
7. E. Fadell, Homotopy groups of configuration spaces and the string problem of Dirac, *Duke Math. J.* **29** (1962) 231–242.
8. E. Fadell and J. Van Buskirk, The braid groups of E^2 and S^2, *Duke Math. J.* **29** (1962) 243–257.
9. `http://www.math.utah.edu/~palais/links.html`
10. J. A. H. Shepperd, Braids which can be plaited with their threads tied together at each end, *Proc. Royal Soc. of London Ser. A* **265**(1321) (1962) 229–244.
11. E. Wigner, On unitary representations of the inhomogeneous Lorentz group, *Math. Ann.* **40**(1) (1939) 149–204.

NOTES

Packing Squares in a Square

IWAN PRATON
Franklin and Marshall College
Lancaster, PA 17604
iwan.praton@fandm.edu

If we put two non-overlapping squares (not necessarily the same size) inside a unit square, then the sum of their circumferences is at most 4, the circumference of the unit square. Apparently this problem was first posed around 1932 by Paul Erdős as a problem for high school students in Hungary [**3**]. It was actually the simplest case of a more general Erdős conjecture: if we put k^2+1 non-overlapping squares inside a unit square, then the total circumference remains at most $4k$ [**4**].

Apparently not much work was done on this conjecture—even the paper by Erdős and Graham [**3**], which starts out by discussing this problem, is mostly about packing *identical unit* squares inside a larger square. In 1995 Erdős, calling it "perhaps undeservedly forgotten" [**2**, as quoted in [**1**]], resurrected the conjecture by offering \$50 for a proof or disproof [**4**]. He and Soifer in [**4**] also considered the more general problem of packing an arbitrary number of squares inside a unit square, not just k^2+1 squares. They provided lower bounds for the total circumference of the squares, and they conjectured that their lower bounds are actually the best possible.

I first learned of this problem from the paper by Campbell and Staton [**1**], who independently also provided lower bounds for the total circumference. They also conjectured that their lower bounds (identical to those of Erdős and Soifer) are the best possible. Naming a conjecture after four people is a bit unwieldly, so we will use initials and call it the ESCS conjecture. In this note we will *not* prove either the original 1932 Erdős conjecture or the seemingly more general ESCS conjecture, but we will show that they are equivalent. If you can prove one of them, then the the other follows.

The problem

Instead of looking at the circumferences of the squares, we will focus on the lengths of their sides, clearly an unimportant change. Therefore put n squares (not necessarily the same size) inside a unit square, so that these squares share no common interior point. Let $e_1, e_2, \ldots, e_n$ denote the side-lengths of these squares. Define $f(n)$ to be the maximum possible value of $\sum_{i=1}^n e_i$. Is there a formula for $f(n)$?

There is a slick proof in [**1**] and [**4**] that $f(k^2)=k$ for all $k \geq 1$: apply the Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(e_1, e_2, \ldots, e_{k^2})$ to get

$$e_1 + e_2 + \cdots + e_{k^2} \leq (1^2 + 1^2 + \cdots + 1^2)^{1/2}(e_1^2 + e_2^2 + \cdots + e_{k^2}^2)^{1/2} \leq k,$$

so $f(k^2) \leq k$. Since the standard $k \times k$ grid reaches this upper bound, we conclude that $f(k^2)=k$.

The original Erdős conjecture is that

$$f(k^2+1) = k \quad \text{for all } k \geq 1.$$

The ESCS conjecture can be stated as follows:

$$f(k^2+2c+1) = k + \frac{c}{k} \quad \text{for all } k > |c|.$$

Here c can be any integer, positive or negative (or zero). For example, the conjecture states that $f(k^2-1) = k - 1/k$ for all $k > 1$. When $c = 0$, the conjecture states that $f(k^2+1) = k$—the original Erdős conjecture. Note that if n is an integer that is not a perfect square, then n lies between two squares of opposite parity, say r^2 and $(r+1)^2$. Hence either $n - r^2$ is odd or $n - (r+1)^2$ is odd, so the conjecture provides values of $f(n)$ for all nonsquare integers n. For example, suppose $n = 22$. Now 22 lies between 16 and 25, and in this case it is $22 - 25 = -3$ that is odd. So we put $k = 5$ and $c = -2$ in the formula, and the conjectured value of $f(22)$ is $5 - 2/5 = 4.6$.

By explicit construction, Erdős and Soifer (also Campbell and Staton) showed that $f(k^2+2c+1) \geq k + c/k$ for all $k > |c|$. Thus in order to prove the conjecture, all we need to do is show that $k + c/k$ is an upper bound for $f(k^2+2c+1)$. This is easier said than done. Instead, we will show that if the formula is correct for one particular value of c, then it must be correct for all values of c. In particular, the values conjectured by ESCS follow from the value conjectured originally by Erdős.

An upper bound

We first show how knowing f at one particular value of its argument can be leveraged into an upper bound for f at a different value.

First put n small squares (in some configuration) inside a unit square. Let A denote the sum of the edge-lengths of the n squares, i.e., $A = \sum_{i=1}^{n} e_i$. Set aside this unit square for the moment. Now take another unit square and divide it into the standard $b \times b$ grid of squares, each with side length $1/b$. Remove an $a \times a$ subsquare, and replace it with our first square, shrunk by a factor of b/a so that it fits inside the $a \times a$ space. FIGURE 1 illustrates this for $n = 7$, $b = 5$, and $a = 3$.

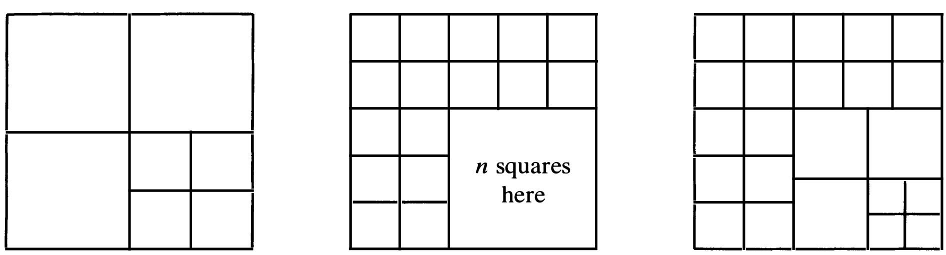


Figure 1 On the left is n squares packed into a unit square. In the center is a $b \times b$ grid with an $a \times a$ subsquare removed. On the right we have filled the $a \times a$ space with a shrunken version of our first square.

We now have a configuration of $b^2 - a^2 + n$ squares inside the unit square. The sum of the side lengths of these squares is $aA/b + (b^2 - a^2)/b$. This is at most $f(b^2 - a^2 + n)$, so we have $aA/b + (b^2-a^2)/b \leq f(b^2 - a^2 + n)$. Rewriting the inequality gives us

$$A \leq a - \frac{b^2}{a} + \frac{b}{a} f(b^2 - a^2 + n).$$

Since our original packing of n squares in the unit square was arbitrary, we conclude that

$$f(n) \le a - \frac{b^2}{a} + \frac{b}{a} f(b^2 - a^2 + n). \tag{1}$$

Thus if we know $f(n + b^2 - a^2)$, then we have an upper bound for $f(n)$. Different values of a and b produce different upper bounds; we will make good use of this fact.

The main result and proof

It's probably worthwhile to state our main result in a formal way. For any integer c, write $P(c)$ for the statement

$$f(k^2 + 2c + 1) = k + \frac{c}{k} \quad \text{for all } k > |c|.$$

We will prove that the truth of $P(c)$ for one value of c implies that $P(c)$ is true for all values of c. In particular, if $P(0)$ is true (the original Erdős conjecture), then all of the $P(c)$'s are true (the ESCS conjecture).

Naturally the proof is by induction on c, but in contrast to the usual case, we need to show not only that $P(c-1) \implies P(c)$ (forward induction), but also that $P(c+1) \implies P(c)$ (backward induction). This situation arises because c can be any integer, including negative integers.

We proceed in two steps. In the first step, we derive a crude upper bound for $f(k^2 + 2c + 1)$ based on equation (1) and the induction assumption.

LEMMA 1. *Suppose $P(c-1)$ is true. Then*

$$f(k^2 + 2c + 1) \le k + \frac{c}{k} + \frac{k+c}{k(k^2-1)} \quad \textit{for all } k > |c|. \tag{2}$$

Similarly, suppose $P(c+1)$ is true. Then

$$f(k^2 + 2c + 1) \le k + \frac{c}{k} + \frac{k-c}{k(k+1)^2} \quad \textit{for all } k > |c|. \tag{3}$$

Proof. We first assume that $P(c-1)$ is true. Suppose $k > |c|$. Put $n = k^2 + 2c + 1$, $a = k - 1$, $b = k$ in equation (1). Then

$$\begin{aligned} b^2 - a^2 + n &= 2k - 1 + k^2 + 2c + 1 = (k+1)^2 + (2c - 1) \\ &= (k+1)^2 + 2(c-1) + 1. \end{aligned}$$

Note that $k + 1 > |c - 1|$, so we can use our hypothesis that $P(c-1)$ is true, i.e., $f(b^2 - a^2 + n) = k + 1 + (c-1)/(k+1) = k + (k+c)/(k+1)$. Thus equation (1) becomes (after some straightforward algebra)

$$\begin{aligned} f(k^2 + 2c + 1) &\le k - 1 - \frac{k^2}{k-1} + \frac{k}{k-1}\left(k + \frac{k+c}{k+1}\right) \\ &= k + \frac{c}{k} + \frac{k+c}{k(k^2-1)}, \end{aligned}$$

as claimed. The proof of equation (3) proceeds similarly, but we use $n = k^2 + 2c + 1$, $a = k + 1$, and $b = k + 2$ in equation (1). ■

We thus now have an upper bound for $f(k^2 + 2c + 1)$ that applies for all $k > |c|$, but it is not quite what we want—it is too big by $(k + c)/(k(k^2 - 1))$. In the second step, we refine the upper bound so that it matches the ESCS lower bound.

LEMMA 2. *Equation* (2) *implies* $P(c)$. *Similarly, equation* (3) *implies* $P(c)$.

Proof. As stated above, it is enough to show that $f(k^2 + 2c + 1) \leq k + c/k$.

First assume equation (2) is true. In equation (1) we let $n = k^2 + (2c + 1)$ as before, but now let $a = k$ and keep b arbitrary. Then $b^2 - a^2 + n = b^2 + (2c + 1)$. Note that we can apply equation (2) to $f(b^2 + 2c + 1)$ since $b > a = k > |c|$. Equation (1) then implies

$$\begin{aligned} f(k^2 + 2c + 1) &\leq k - \frac{b^2}{k} + \frac{b}{k} f(b^2 + (2c + 1)) \\ &\leq k - \frac{b^2}{k} + \frac{b}{k}\left(b + \frac{c}{b} + \frac{b + c}{b(b^2 - 1)}\right) \\ &= k + \frac{c}{k} + \frac{b + c}{k(b^2 - 1)}. \end{aligned}$$

This is true for any value of $b > k$. Now let $b \to \infty$. We get

$$f(k^2 + 2c + 1) \leq k + c/k.$$

which is exactly what we want.

The other half of the lemma is proved similarly. Details are left to the reader. ∎

Putting lemmas (1) and (2) together, we get our main theorem.

THEOREM. *If $P(c)$ is true for one value of c, then it is true for all values of c.*

One final note on this topic. Looking carefully at the proof of Lemma (2), we see that in order to prove the ESCS conjecture, it suffices to show that $f(k^2 + 2c + 1) = k + c/k + \epsilon(k)$, where $k\epsilon(k) \to 0$ as $k \to \infty$. Unfortunately, in order to do this it is probably necessary to investigate in detail the placement of the n squares inside the unit square.

REFERENCES

1. C. Campbell and W. Staton, A square packing problem of Erdős, *Amer. Math. Monthly* **112** (2005) 165–167.
2. P. Erdős, Some of my favorite problems in number theory, combinatorics, and geometry, *Resenhas* **2** (1995) 165–186.
3. P. Erdős and R. Graham, On packing squares with equal squares, *J. Combin. Th. Ser. A* **19** (1975) 119–123.
4. P. Erdős and A. Soifer, Squares in a square, *Geombinatorics* **4** (1995) 110–114.

Why Is the Sum of Independent Normal Random Variables Normal?

BENNETT EISENBERG
Lehigh University
Bethlehem, PA 18015
BE01@lehigh.edu

ROSEMARY SULLIVAN
West Chester University
West Chester, PA 19383
rsullivan@wcupa.edu

The fact that the sum of independent normal random variables is normal is a widely used result in probability. Two standard proofs are taught, one using convolutions and the other moment generating functions, but neither gives much insight into why the result is true. In this paper we give two additional arguments for why the sum of independent normal random variables should be normal.

The convolution proof

The first standard proof consists of the computation of the convolution of two normal densities to find the density of the sum of the random variables. Throughout this article we assume that our normal random variables have mean 0 since a general normal random variable can be written in the form $\sigma Z + \mu$, where Z is standard normal and μ is a constant. One then finds the convolution of two normal densities to be

$$\int_{-\infty}^{\infty} \frac{\exp\left(-\frac{(y-x)^2}{2\sigma_1^2}\right)}{\sqrt{2\pi}\sigma_1} \frac{\exp\left(-\frac{x^2}{2\sigma_2^2}\right)}{\sqrt{2\pi}\sigma_2} dx = \frac{\exp\left(-\frac{y^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}.$$

The computation is messy and not very illuminating even for the case of mean zero random variables.

The moment generating proof

The calculation of convolutions of probability distributions is not easy, so proofs using moment generating functions are often used. One uses the fact that the moment generating function of a sum of independent random variables is the product of the corresponding moment generating functions. Products are easier to compute than convolutions.

We have that the moment generating function of a mean zero normal random variable X with variance σ^2 is

$$M_X(t) = \int_{-\infty}^{\infty} \exp(tx) \frac{\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sqrt{2\pi}\,\sigma} dx = \exp\left(\frac{t^2\sigma^2}{2}\right).$$

Thus, if X_1 and X_2 are independent, mean zero, normal random variables with variances σ_1^2 and σ_2^2, respectively, then

$$M_{X_1+X_2}(t) = \exp\left(\frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}\right).$$

We see that the product of the moment generating functions of normal random variables is also the moment generating function of a normal random variable. The result then follows from the uniqueness theorem for moment generating functions, i.e., the fact that the moment generating function of a random variable determines its distribution uniquely.

This argument is a little more illuminating. At least we can see what is happening in terms of the moment generating functions. Of course, the fact that the moment generating function of a normal random variable takes this nice form is not obvious. We also must use the uniqueness theorem to make the proof complete.

The rotation proof

The geometric proof that we now present has the advantage that it is more visual than computational. It is elementary, but requires a bit more sophistication than the earlier proofs.

We begin with two independent standard normal random variables, Z_1 and Z_2. The joint density function is

$$f(z_1, z_2) = \frac{\exp(-\frac{1}{2}(z_1^2 + z_2^2))}{2\pi},$$

which is rotation invariant (see FIGURE 1). That is, it has the same value for all points equidistant from the origin. Thus, $f(T(z_1, z_2)) = f(z_1, z_2)$, where T is any rotation of the plane about the origin.

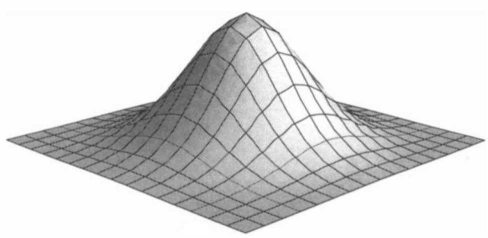


Figure 1 $f(z_1, z_2) = \frac{\exp\left(-\frac{z_1^2+z_2^2}{2}\right)}{2\pi}$ is rotation invariant.

It follows that for any set A in the plane $P((Z_1, Z_2) \in A) = P((Z_1, Z_2) \in TA)$, where T is a rotation of the plane. Now if X_1 is normal with mean 0 and variance σ_1^2 and X_2 is normal with mean 0 and variance σ_2^2, then $X_1 + X_2$ has the same distribution as $\sigma_1 Z_1 + \sigma_2 Z_2$. Hence

$$P(X_1 + X_2 \leq t) = P(\sigma_1 Z_1 + \sigma_2 Z_2 \leq t) = P((Z_1, Z_2) \in A),$$

where A is the half plane $\{(z_1, z_2) \mid \sigma_1 z_1 + \sigma_2 z_2 \leq t\}$. The boundary line $\sigma_1 z_1 + \sigma_2 z_2 = t$ lies at a distance $d = \frac{|t|}{\sqrt{\sigma_1^2+\sigma_2^2}}$ from the origin.

It follows that the set A can be rotated into the set

$$TA = \left\{ (z_1, z_2) | \, z_1 < \frac{t}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right\}.$$

(See FIGURE 2 for the case $t > 0$ and FIGURE 3 for the case $t < 0$.)

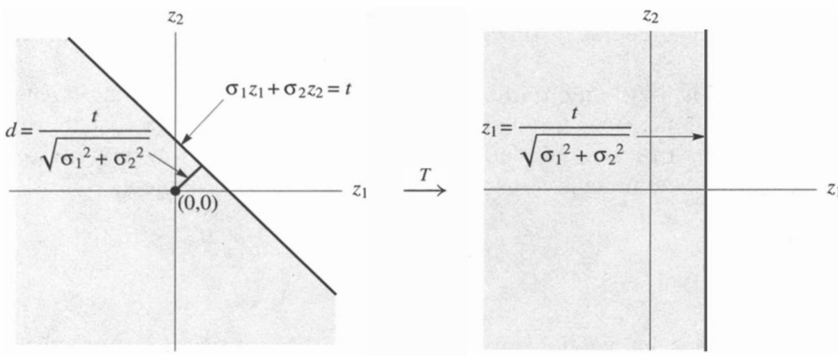


Figure 2 The half plane $\sigma_1 z_1 + \sigma_2 z_2 \leq t,\ t > 0$ is rotated into the half plane $z_1 \leq \frac{t}{\sqrt{\sigma_1^2+\sigma_2^2}}$.

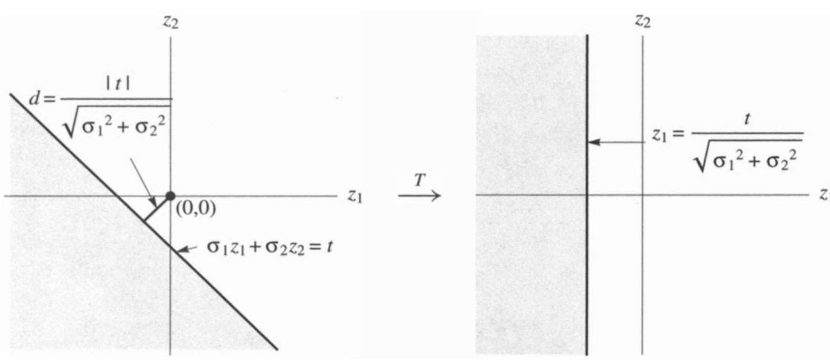


Figure 3 The half plane $\sigma_1 z_1 + \sigma_2 z_2 \leq t,\ t < 0$ is rotated into the half plane $z_1 \leq \frac{t}{\sqrt{\sigma_1^2+\sigma_2^2}}$.

Thus $P(X_1 + X_2 < t) = P(\sqrt{\sigma_1^2 + \sigma_2^2} Z_1 < t)$. It follows that $X_1 + X_2$ is normal with mean 0 and variance $\sigma_1^2 + \sigma_2^2$. This completes the proof.

In all the probability texts that we have surveyed, we have only found one [**2**, pp. 361–363] with an approach based on the rotation invariance of the joint normal density.

Generalizing the rotation argument

The next proposition follows from the same rotation argument used to show that $\sigma_1 Z_1 + \sigma_2 Z_2$ is equal to $\sqrt{\sigma_1^2 + \sigma_2^2} Z$ in the normal case.

PROPOSITION. *Assume that X and Y are random variables with rotation invariant joint distribution and X has density $f_X(x)$. Let $Z = aX + bY$. Then Z has density*

$$f_Z(z) = \frac{1}{\sqrt{a^2+b^2}} f_X\left(\frac{z}{\sqrt{a^2+b^2}}\right).$$

When X is normal, this implies that $aX + bY$ is normal. Another example occurs when (X, Y) is uniformly distributed over the unit disc. Then X has density $f(x) = \frac{2}{\pi}\sqrt{1-x^2}$ for $-1 \le x \le 1$. It follows that $aX + bY$ has density $f_c(x) = \frac{2}{c\pi}\sqrt{1-\frac{x^2}{c^2}}$, for $-c \le x \le c$, where $c = \sqrt{a^2+b^2}$. We note in this example that X and Y are not independent. It is a well known result [**1**, p. 78] that if X and Y are independent with rotation invariant joint density, then X and Y must be mean 0 normal random variables. Thus we cannot use this method to find the density of $aX + bY$ for independent X and Y except in the case where X and Y are normal.

The algebraic proof

It is possible to give a simple, plausible algebraic argument as to why the sum of independent normal random variables is normal if one is allowed to assume the central limit theorem. The central limit theorem implies that if $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean 0 and variance 1, then

$$P\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \le t\right) \to P(Z_1 \le t),$$

and

$$P\left(\frac{X_{n+1} + \cdots + X_{2n}}{\sqrt{n}} \le t\right) \to P(Z_2 \le t),$$

where Z_1 and Z_2 are independent, standard normal random variables. Furthermore,

$$P\left(\frac{X_1 + \cdots + X_{2n}}{\sqrt{2n}} \le t\right) \to P(Z_3 \le t),$$

where Z_3 is also standard normal. Since

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} + \frac{X_{n+1} + \cdots + X_{2n}}{\sqrt{n}} = \frac{X_1 + \cdots + X_{2n}}{\sqrt{n}} = \sqrt{2}\,\frac{X_1 + \cdots + X_{2n}}{\sqrt{2n}},$$

it would seem reasonable that $Z_1 + Z_2$ has the same distribution as $\sqrt{2}\,Z_3$, i.e., $Z_1 + Z_2$ is normal with mean 0 and variance 2. This argument can be made rigorous using facts about convergence in distribution of random variables.

A similar argument using the fact that

$$P\left(\frac{X_1 + \cdots + X_{[\sigma n]}}{\sqrt{n}} \le t\right) \to P(\sigma Z \le t)$$

would show why the sums of general independent normal random variables must be normal.

This algebraic argument is a nice conceptual argument for showing why the sum of independent normal random variables must be normal, but it assumes the central limit theorem, which is not obvious or easy to prove, as well as facts about convergence in distribution.

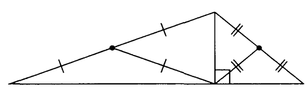
The rotation proof seems better to us than the others since it is elementary, self contained, conceptual, uses clever geometric ideas, and requires little computation. Whether it would give more insight to the average student is difficult to say. Nevertheless, with all of this in its favor, it ought to be more widely taught.

REFERENCES

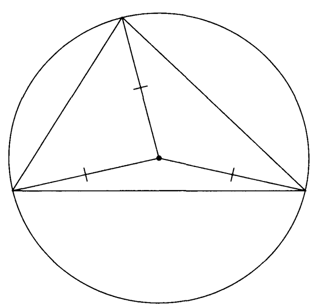
1. W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. II*, John Wiley Sons, New York, 1971.
2. J. Pitman, *Probability*, Springer-Verlag, New York, 1993.

Proof Without Words: Isosceles Dissections

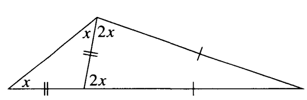
Every triangle can be dissected into four isosceles triangles:

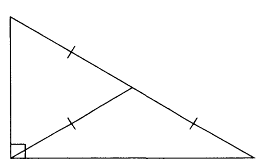


Every acute-angled triangle can be dissected into three isosceles triangles:



A triangle can be dissected into two isosceles triangles if and only if one of its angles is three times another or if the triangle is right angled:





REFERENCE

Angel Plaza, Proof without words: Every triangle can be subdivided into six isosceles triangles, this MAGAZINE **80** (2007).

DES MACHALE
Department of Mathematics
University College, Cork, Ireland

A Converse to a Theorem on Linear Fractional Transformations

XIA HUA
Massachusetts Institute of Technology
Cambridge, MA 02139
xia_hua@mit.edu

It is a well-known theorem in introductory complex analysis that a linear fractional transformation (also called Möbius transformation and bilinear transformation) is a bijection of the extended complex plane that maps circles and lines onto circles and lines [**1**]. It is easy to verify that the complex conjugate of a linear fractional transformation is also such a bijection. An interesting question is what we can say about an arbitrary bijection that maps circles and lines onto circles and lines. Is it necessary for such a map to be either a linear fractional transformation or the complex conjugate of a linear fractional transformation? The answer is YES. Note that such a map is not even assumed to be continuous.

The converse theorem and its proof

Let us first state the theorem in a formal way. We will use the symbol $\overline{\mathbb{C}}$ to denote the set of extended complex numbers $\mathbb{C} \cup \{\infty\}$.

THEOREM. *If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a bijection that maps every circle or line onto a circle or line, then f is either a linear fractional transformation or the complex conjugate of a linear fractional transformation.*

Proof. First we can reduce to the case where $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$. To see this, suppose $f(0) = \omega_1$, $f(1) = \omega_2$, $f(\infty) = \omega_3$ where ω_1, ω_2, ω_3 are three distinct complex numbers. Then there is a unique linear fractional transformation ϕ such that $\phi(\omega_1) = 0$, $\phi(\omega_2) = 1$, and $\phi(\omega_3) = \infty$ [**1**]. Hence ϕf is a bijection which satisfies $\phi f(\omega_1) = 0$, $\phi f(\omega_2) = 1$, and $\phi f(\omega_3) = \infty$. If we can show that ϕf is a linear fractional transformation, then using the facts that linear fractional transformations are invertible and compositions of linear fractional transformations are again linear fractional transformations we can conclude that $f = \phi^{-1}\phi f$ is also a linear fractional transformation.

So from now on we assume 0, 1, and ∞ are fixed points. We want to prove that f is the identity map or its complex conjugate.

The following facts will be used frequently:

(1) Since f fixes ∞, lines are mapped to lines and circles to circles.

(2) If two lines are parallel, then they intersect at ∞ so their images are also parallel.

(3) If a line is tangent to a circle, so is the image of this line to the image of the circle.

(4) Since f fixes 0, 1, and ∞, the real axis maps to itself.

It follows from (2) and (4) that horizontal lines are mapped to horizontal lines.

We can show that vertical lines are mapped to vertical lines. First the y-axis is mapped onto itself. To see this, take the circle C_0 centered at $\frac{1}{2}$ with radius $\frac{1}{2}$. The tangent lines to this circle at 0 and 1, denoted by T_0, T_1, are parallel to each other so by (2) their images are parallel. Also by (3), the images of T_0, T_1 are tangent to the image of the circle. Since 0 and 1 are fixed, C_0 is mapped to a circle passing through 0 and 1. If its center were not on the real axis, then the tangent lines at 0 and 1 would intersect each other which is a contradiction. Since there is a unique circle with its center on the real axis and passing through 0, 1, C_0 is mapped onto itself. Consequently T_0 is mapped onto itself. Consider an arbitrary vertical line, say $T_a : x = a \in \mathbf{R}$. Since T_a is parallel to T_0, $f(T_a)$ is parallel to $f(T_0)$. But $f(T_0)$ is a vertical line so $f(T_a)$ is also a vertical line.

Now we use induction to show that

$$f(n) = n, \quad \forall n \in \mathbf{N}.$$

This is true for 0 and 1. Suppose it is true for all $n < k + 1$. Then for $n = k + 1$, we look at the circle C_k centered at k with radius 1. Since k is fixed by f, the vertical line $x = k$ is mapped onto itself. The horizontal line $y = 1$ is tangent to C_k at the point $(k, 1)$. Consequently the image of this line, which is also horizontal, is tangent to $f(C_k)$ at $f(k, 1)$, which is on the line $x = k$. This implies $f(C_k)$ is a circle symmetric about the vertical line $x = k$. Thus $f(k + 1)$ and $f(k - 1)$, being the points of intersection of $f(C_k)$ with the x-axis, are symmetric about the line $x = k$. Since the points $k - 1$ and k are fixed, $k + 1$ is also fixed. Using similar arguments we can show that

$$f(n) = n, \quad \forall n \in \mathbf{Z}.$$

We can further prove that

$$f\left(\frac{2n-1}{2}\right) = \frac{2n-1}{2}, \quad \forall n \in \mathbf{Z}.$$

To see this, fix $n \in \mathbf{Z}$ and consider the circle C centered at $\frac{2n-1}{2}$ with radius $\frac{1}{2}$. Since the points $n - 1$ and n are fixed, $x = n$ and $x = n - 1$ are vertical tangents to $f(C)$. It follows that $f(C) = C$ as they must have the same center and radius. Consequently $y = \frac{1}{2}$, which is the horizontal tangent line to C at $(\frac{2n-1}{2}, \frac{1}{2})$, is mapped to itself or to $y = -\frac{1}{2}$. This means

$$f\left(\frac{2n-1}{2}, \frac{1}{2}\right) = \left(\frac{2n-1}{2}, \frac{1}{2}\right) \quad \text{or} \quad \left(\frac{2n-1}{2}, -\frac{1}{2}\right).$$

In either case it follows that the vertical line through $(\frac{2n-1}{2}, \frac{1}{2})$ maps to itself so $f(\frac{2n-1}{2}) = \frac{2n-1}{2}$.

Similar arguments can be used to prove that

$$f\left(\frac{2n-1}{2^k}\right) = \frac{2n-1}{2^k}, \quad \forall n \in \mathbf{Z},\ \forall k \in \mathbf{N}.$$

Note that the set $A = \{\frac{2n-1}{2^k}, n \in \mathbf{Z}, k \in \mathbf{N}\}$ is dense in $\mathbf{R}$. If there is a point a on the real line such that $f(a) \neq a$, we can find n and k such that $\frac{2n-1}{2^k}$ lies between the points $f(a)$ and a. Draw a circle $C_{n,k}$ with its center on the real axis and passing through the points 0, $\frac{2n-1}{2^k}$. Then we see that this circle intersects one and only one of the two vertical lines, $x = a$ and $x = f(a)$. Without loss of generality, suppose that $C_{n,k}$ intersects $x = a$ but not $x = f(a)$. Note that $C_{n,k}$ is mapped onto itself since both 0 and $\frac{2n-1}{2^k}$ are fixed. This means

$$C_{n,k} \cap (x = a) \neq \emptyset,$$

but

$$f(C_{n,k}) \cap (x = f(a)) = C_{n,k} \cap (x = f(a)) = \emptyset.$$

which is a contradiction. Hence the real line is fixed.

Finally we consider the imaginary axis. Look at the standard unit circle. Since -1 and 1 are fixed, the unit circle is mapped onto itself so the horizontal tangent at (0, 1) is mapped to either the horizontal tangent at (0, 1) or the horizontal tangent at $(0, -1)$. Hence either $f(0, 1) = (0, 1)$ or $f(0, 1) = (0, -1)$. If $f(0, 1) = (0, 1)$, then $0, i$, and ∞ are fixed. Our argument showing that the real axis is fixed applies to the imaginary axis. If $f(0, 1) = (0, -1)$, then the conjugate of f, denoted by f^*, fixes 0, i, and ∞. It follows that f^* fixes both the real and imaginary axes.

For an arbitrary point (a, b) in the complex plane, there are two cases depending on whether f fixes the imaginary axis or its conjugate f^* fixes the imaginary axis. In the case where f fixes the imaginary axis, the vertical line $x = a$ and the horizontal line $y = b$ are both mapped to themselves so their intersection is fixed, that is $f(a, b) = (a, b)$. Since (a, b) is an arbitrary point, it follows that f is the identity. In the case where f^* fixes the imaginary axis, a similar argument shows that f^* is the identity. Therefore f is the conjugate of the identity. ■

Conclusion

Motivated by the theorem that a linear fractional transformation is a bijection that maps circles and lines onto circles and lines, we proposed and proved a converse theorem which states that any bijective (not necessarily continuous) function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that maps every circle or line onto a circle or line is either a linear fractional transformation or the complex conjugate of a linear fractional transformation.

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REFERENCE

1. Donald Sarason, *Notes on complex function theory*, Henry Helson, Berkeley, CA, USA, 1994.

Sublimital Analysis

THOMAS Q. SIBLEY
St. John's University
Collegeville, MN 56321
tsibley@csbsju.edu

Limits of subsequences play a small supporting role in analysis. (See, for example, the Bolzano-Weierstrass Theorem.) However, in the typical undergraduate course we never seem to care what the limits actually are, suggesting that these "sublimits" might not deserve star billing. The article [**3**] by Zheng and Cheng in the references does consider such "sublimits" in a particular setting. This article takes a closer look at subsequences and their limits more generally. I owe a disclosure to those readers who connected the "sublimital" of the title with the word "subliminal." While "sublimits" may well be hidden in the original sequence, they aren't placed there to send subconscious messages. Instead we should think of them as enticements to mathematical exploration.

EXAMPLE 1. Let (b_n) be the alternating sequence given by $b_n = (-1)^n(\frac{n-1}{n})$. The first few terms are $\frac{0}{1}, \frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \ldots$. This sequence diverges, but it has subsequences converging to two "sublimits":

$$\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} \frac{2k-1}{2k} = 1 \quad \text{and} \quad \lim_{k\to\infty} b_{2k-1} = \lim_{k\to\infty} \frac{-(2k-2)}{2k-1} = -1.$$

DEFINITION. Given a sequence (a_n) of real numbers, a real number s is a *sublimit* of (a_n) if and only if there is some subsequence (a_{n_k}) of the original sequence such that $\lim_{k\to\infty} a_{n_k} = s$. Denote the set of all sublimits of a sequence (a_n) by $S(a_n)$.

EXAMPLE 1 (CONTINUED). For the sequence (b_n), where $b_n = (-1)^n(\frac{n-1}{n})$ as given earlier, $S(b_n) = \{1, -1\}$. The reader is invited to show (b_n) has no other sublimits.

Remark. If a sequence (c_n) converges to a limit l, then $S(c_n) = \{l\}$ since l is the only possible sublimit.

EXAMPLE 2. The sequence (d_n) given by $d_n = n$ has no sublimits and $S(d_n) = \varnothing$.

EXAMPLE 3. The set $\mathbb{Q}$ of rationals is countable so there is a sequence (q_n) listing all of $\mathbb{Q}$. Then every real number r is a sublimit of (q_n). To see this, note that for each $\varepsilon = \frac{1}{k} > 0$ there are infinitely many rationals in the open interval $(r - \varepsilon, r + \varepsilon)$. So for all $k \in \mathbb{N}$, we can choose $q_{n_k} \in \mathbb{Q}$ such that $|q_{n_k} - r| < \frac{1}{k}$ and $n_k < n_{k+1}$. Thus the subsequence (q_{n_k}) converges to r and $S(q_n) = \mathbb{R}$.

In Example 3 each real number has its own subsequence, which the usual notation (q_{n_k}) can't indicate. The following notation overcomes that lack and will be useful in the proof of Theorem 1.

DEFINITION. Given a sequence (a_n) and a set of its subsequences indexed by K, for $k \in K$ let $(a_n(k))$ denote the subsequence with index k and let $a_n(k, i)$ denote the ith term of $(a_n(k))$.

Example 3 naturally leads to the question "Given any set S of real numbers is there a sequence whose set of sublimits is S?" The answer, in a word, is "no." Our goal is to characterize the possible sets of sublimits. At the end we'll generalize this question to metric spaces.

In looking for ways to describe possible sets of sublimits we might well start with instances where subsequences appear in analysis courses. (See, for example, the text [1] by Abbott for definitions of terms used in this paragraph along with more on the theorems.) The Bolzano-Weierstrass theorem states that every bounded sequence has a convergent subsequence. So perhaps bounded sets play a role. However, the set of sublimits in Example 3 is definitely not bounded, so that property can't be part of the characterization of sets of sublimits. Another common role of subsequences is in the definition of sequentially compact. The Heine-Borel theorem informs us that sets are compact if and only if they are closed and bounded. Also, limits and sublimits are related to limit points, which appear in the definition of closed sets. So perhaps the characterization of sets of sublimits relates to closed sets. The sets of sublimits in Examples 1, 2 and 3 are, indeed, closed. Theorem 1 below confirms that all sets of sublimits are closed.

THEOREM 1. *Let (a_n) be any sequence of real numbers. Then $S(a_n)$, its set of sublimits, is a closed set.*

FIGURE 1 explains the idea behind the proof. The right hand column is a sequence (s_k) of sublimits of the sequence (a_n) and its limit L. Each of the sublimits s_k has a subsequence $(a_n(k))$ converging to s_k. We need to create a new subsequence $(a_n(L))$ converging to L. FIGURE 1 suggests choosing the "diagonal" subsequence $(a_n(k, k))$. The reader is invited to construct an example where such a diagonal subsequence fails to converge to L. In the proof we shall generate a more sophisticated subsequence using the Axiom of Choice.

$$\begin{array}{cccccc}
a_n(1,1) & a_n(1,2) & a_n(1,3) & \dots & & s_1 \\
a_n(2,1) & a_n(2,2) & a_n(2,3) & \dots & & s_2 \\
a_n(3,1) & a_n(3,2) & a_n(3,3) & \dots & & s_3 \\
\vdots & \vdots & \vdots & & & \vdots \\
a_n(k,1) & a_n(k,2) & \dots & a_n(k,k) & \dots & s_k \\
\vdots & \vdots & & & \ddots & \vdots \\
 & & & & & L
\end{array}$$

Figure 1

Proof of Theorem 1. Let (a_n) be any sequence and, in order to show $S(a_n)$ is closed, let L be any limit point of $S(a_n)$. Then there is a sequence (s_k) such that for each $k \in \mathbb{N}$, $s_k \in S(a_n)$, $s_k \neq L$ and $\lim_{k\to\infty} s_k = L$. Because each s_k is a sublimit of (a_n), there is a subsequence $(a_n(k))$ such that $\lim_{i\to\infty} a_n(k, i) = s_k$. (See FIGURE 1.) We need to build a new subsequence $(a_n(L))$ converging to L in order to show $S(a_n)$ is closed.

For $i \in \mathbb{N}$, let $j(i)$ be the smallest subscript such that for $|s_{j(i)} - L| < \frac{1}{i}$ and let $a_n(j(i), h(i))$ be the first term of (a_n) such that $|a_n(j(i), h(i)) - s_{j(i)}| < \frac{1}{i}$. That is, we choose the sublimit $s_{j(i)}$ to be close to our ultimate limit L and in turn choose the term $a_n(j(i), h(i))$ to be close to $s_{j(i)}$. Thus $a_n(j(i), h(i))$ must be fairly close to L. More precisely, $|a_n(j(i), h(i)) - L| < \frac{2}{i}$. Note that without uniform convergence of the subsequences $(a_n(k))$ we need the Axiom of Choice to ensure the existence of all of the $h(i)$.

We are now ready to define our subsequence $(a_n(L))$ recursively. We take $a_n(L, 1) = (a_n(j(1), h(1))$. Given $a_n(L, w)$, define $a_n(L, w + 1)$ to be the first term $a_n(j(i), h(i))$ such that $i \geq (w + 1)$ and $a_n(j(i), h(i))$ has a larger index in the original sequence (a_n) than $a_n(L, w)$ has. Since there are infinitely many terms $a_n(j(i), h(i))$, there are terms satisfying these conditions. Then we have $|a_n(L, w) - L| < \frac{2}{w}$ and the subsequence $a_n(L, w)$ converges to L. Hence L is a sublimit of (a_n) and $S(a_n)$ is closed. ■

Now that we know the set of sublimits is closed, we turn the situation around and show in Theorem 2 that every closed set of reals is a set of sublimits. The proof of Theorem 2 is more involved than the first proof since it requires constructing a sequence to fit a given closed set and ensuring that no extraneous sublimits sneak in.

THEOREM 2. *For any closed subset F of $\mathbb{R}$ there is a sequence (a_n) such that $S(a_n) = F$.*

Proof. We may assume that the closed set F is non-empty since otherwise we could use the sequence of Example 2. To simplify notation, we further assume $0 \in F$. (If $0 \notin F$ but $a \in F$, we adapt the following construction by adding a throughout.)

To approximate every element of F we first define a collection of intervals $I_{i,j}$, where $i \in \mathbb{N} \cup \{0\}$ and $1 \le j \le 2 \cdot 4^i$. (See FIGURE 2.) Define $I_{i,j} = [\,-2^i + (j-1)(2^{-i}),\ -2^i + (j)(2^{-i})]$ for $i \ge 0$ and $1 \le j \le 2 \cdot 4^i$. Note that $\bigcup_{j=1}^{2\cdot 4^i} I_{i,j} = [-2^i, 2^i]$. Thus each time i increases, the family of intervals $\{I_{i,j} : 1 \le j \le 2 \cdot 4^i\}$ covers an interval twice as long with intervals half as long.

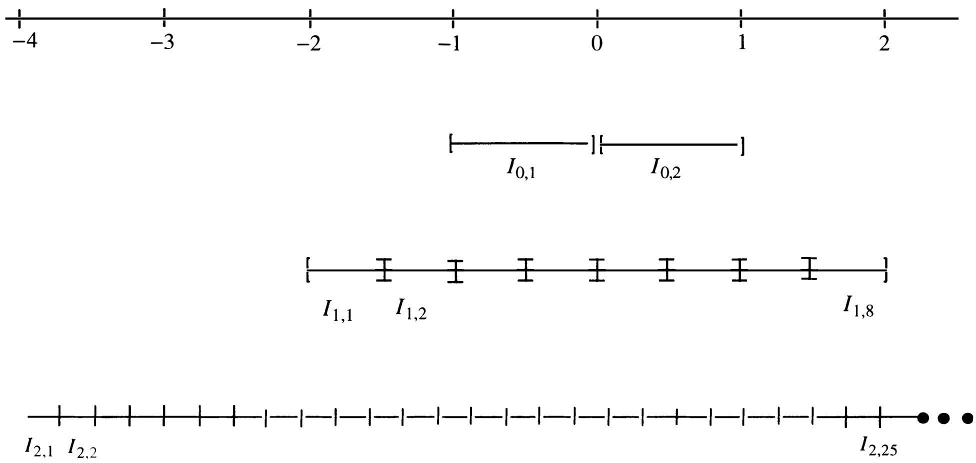


Figure 2 The intervals I_{ij}.

For each interval $I_{i,j}$ we choose a number $a_{i,j}$, which may or may not be in the interval. If $F \cap I_{i,j}$ is non-empty, we let $a_{i,j}$ be the midpoint of $I_{i,j}$. Otherwise, $a_{i,j} = 0$. (FIGURE 3 illustrates the numbers $a_{i,j}$ for a specific set F and the intervals $I_{i,j}$.) We use the lexicographic order on the numbers $a_{i,j}$ to obtain a sequence. That is, $a_{i,j}$ comes before $a_{n,k}$ if and only if $i < n$ or ($i = n$ and $j < k$). [The sequence starts off $a_{0,1}$, $a_{0,2}$, $a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{1,8}, a_{2,1}$, etc.] Let b_n be the nth term of the $a_{i,j}$ using this ordering.

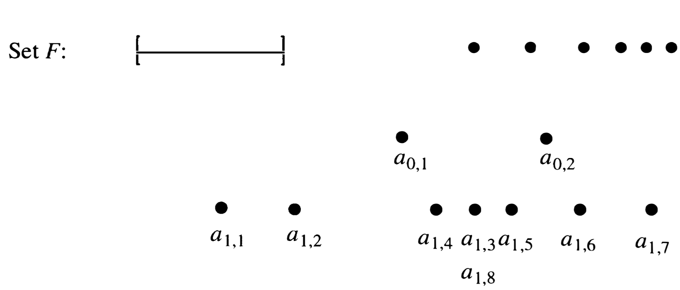


Figure 3 A given closed set F and its corresponding terms a_{ij}. The sequence starts out $-\frac{1}{2}, \frac{1}{2}, -1\frac{3}{4}, -1\frac{1}{4}, 0, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 1\frac{1}{4}, 0, \dots$

Claim. F is the set of sublimits of (b_n). First we show that if $x \in F$, then $x \in S(b_n)$, and then we show the converse. Let $x \in F$. There is $n \in \mathbb{N}$ such that $|x| \le 2^n$. For $i \ge n$, there is one (or possibly two) choices of j such that $x \in I_{i,j}$. For these i and j, we see that $|x - a_{i,j}| \le 2^{-i-1}$ because $a_{i,j}$ is the midpoint of an interval of length 2^{-i}. Thus we can form a subsequence $(b_n(x))$ from these $a_{i,j}$ and $(b_n(x))$ converges to x. So x is a sublimit of (b_n).

Suppose $y \notin F$. Since F is closed, there is $\epsilon > 0$ such that the interval $(y - \epsilon, y + \epsilon)$ is disjoint from F. However, that doesn't mean that each $a_{i,j}$ must be at least

ϵ away from y. If there is $w \in I_{i,j} \cap F$, then $a_{i,j}$, the midpoint of $I_{i,j}$, satisfies $|w - a_{i,j}| \leq 2^{-i-1}$. Since $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$. Hence for $i \geq n$ and any j, the closest $a_{i,j}$ could be to y is $|y - a_{i,j}| \geq |y - w| - |w - a_{i,j}| \geq 2^{-n-1}$. Thus no subsequence of (b_n) can converge to y and $S(b_n) = F$, as claimed. ■

The proof of Theorem 1 generalizes readily to any metric space. The proof of Theorem 2 generalizes to $\mathbb{R}^n$ by replacing the intervals $I_{i,j}$ with n-dimensional "boxes." However, Example 4 below shows that there are metric spaces for which Theorem 2 fails.

EXAMPLE 4. Let $\mathbb{F}$ be the set of all real functions $f : \mathbb{R} \to [0, 1]$ and define a metric d on $\mathbb{F}$ by $d(f, g) = \sup |f(x) - g(x)|$. The whole space is closed, as is any metric space. However, we will show that no sequence of functions has the whole space as its set of sublimits. Let (f_n) be any sequence in $\mathbb{F}$. Consider the new function $f : \mathbb{R} \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{N} \\ f_n(n) + 0.5 & \text{if } x = n \text{ and } f_n(n) \leq 0.5 \\ f_n(n) - 0.5 & \text{if } x = n \text{ and } f_n(n) > 0.5 \end{cases}$$

Then $d(f, f_n) \geq |f(n) - f_n(n)| = 0.5$. Thus no subsequence of (f_n) can approach the function f. The reader is invited to determine some of the many other closed subsets of $\mathbb{F}$ that are not sets of sublimits.

The key to generalizing Theorem 2 successfully is the existence of a subset like the midpoints of the intervals $I_{i,j}$, which is a countable, dense subset of $\mathbb{R}$. A subset S of a metric space X is *dense* in X if and only if the closure of S is X. Equivalently, S is dense in X if and only if for every $x \in X$ there is a sequence of elements of S converging to x. For a sequence to have the whole space as its set of limit points, the sequence as a set must be dense in the space. Since sequences have countably many terms, only spaces with countable dense subsets can be candidates to generalize Theorem 2. Theorem 3 below assures us they do. The reader is encouraged to prove Theorem 3 assuming the following fact, proven in Kuratowski [**2**, p. 156]: If a metric space has a countable dense subset, then every subset of it does too. The reader should also consider why we need to require $F \neq \varnothing$ in this theorem.

THEOREM 3. *If a metric space X has a countable dense subset and F is a non-empty closed set in X, then there is a sequence (a_n) whose set $S(a_n)$ of sublimits is F.*

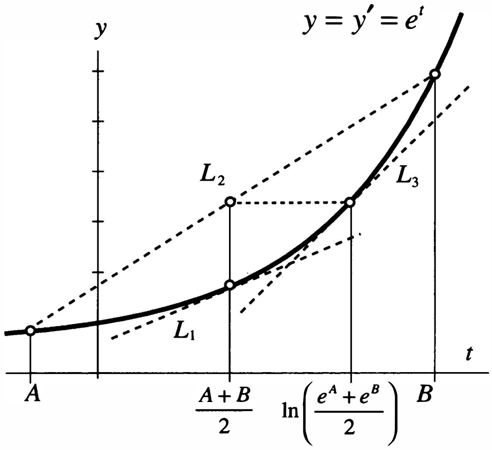
The close connection of sublimits with the deeper idea of closed sets helps explain why sublimits have not been studied more extensively for their own sake.

REFERENCES

1. Stephen Abbott, *Understanding Analysis*, Springer, New York, 2001.
2. Kazimierz Kuratowski, *Introduction to Set Theory and Topology*, Addison Wesley, Reading, MA, 1962.
3. Sining Zheng and Jiangchen Cheng, Density of the images of integers under continuous functions with irrational periods, this MAGAZINE **72** (1999) 402–404.

Proof Without Words: Exponential Inequalities

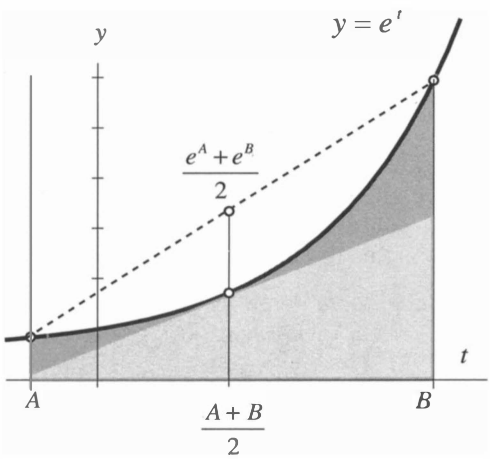
$$A < B \Rightarrow e^{\frac{A+B}{2}} < \frac{e^B - e^A}{B - A} < \frac{e^A + e^B}{2}$$



First semester calculus:

$$m(L_1) < m(L_2) < m(L_3)$$

$$e^{\frac{A+B}{2}} < \frac{e^B - e^A}{B - A} < \frac{e^A + e^B}{2}$$



Second semester calculus:

$$e^{\frac{A+B}{2}}(B-A) < \int_A^B e^t dt < \frac{e^A + e^B}{2}(B-A)$$

$$e^{\frac{A+B}{2}} < \frac{e^B - e^A}{B - A} < \frac{e^A + e^B}{2}$$

ÁNGEL PLAZA
ULPGC, 35017-Las Palmas G.C., Spain
aplaza@dmat.ulpgc.es

PROBLEMS

PROPOSALS

To be considered for publication, solutions should be received by May 1, 2009.

1806. *Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.*

The intersection of the ellipsoid $x^2 + y^2 + \frac{z^2}{c^2} = 1$ and the plane $x + y + cz = 0$ is an ellipse. For $c > 1$, find the value of c for which the area of the ellipse is maximal.

1807. *Proposed by Lenny Jones, Shippensburg University, Shippensburg, PA.*

Let P be a polynomial with integer coefficients and let s be an integer such that for some positive integer n, $s^{n+1}P(s)^n$ is a positive zero of P. Prove that $P(2) = 0$.

1808. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Let α and β be positive real numbers with $\alpha\beta = \pi$, and let y be a real number. Prove that

$$\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\alpha k} \cos(\alpha y k) = \frac{1}{\alpha} \sum_{j=-\infty}^{\infty} \frac{1}{1 + (y + 2\beta j)^2}.$$

1809. *Proposed by Cosmin Pohoata, Tudor Vianu National College of Informatics, Bucharest, Romania.*

Let M be a point on the circumcircle of triangle ABC and lying on the arc BC that does not contain A. Let I be the incenter of ABC, and let E and F be the feet of the

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. Each solution should begin on a separate sheet.

Solutions and new proposals should be mailed to Elgin Johnston, Problems Editor, Department of Mathematics, Iowa State University, Ames IA 50011, or mailed electronically (ideally as a LaTeX file) to ehjohnst@iastate.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

perpendiculars from I to lines MB and MC, respectively. Prove that the value of

$$\frac{IE + IF}{AM}$$

is independent of the position of M.

1810. *Proposed by Greg Oman, Otterbein College, Westerville, OH.*

Let R be a ring. For elements $x, y \in R$ we say x divides y on the right if and only if there is a $z \in R$ with $xz = y$. (We denote this by $x|_r y$.) An element $p \in R$ is a right prime if and only if whenever $p|_r xy$, then either $p|_r x$ or $p|_r y$. Prove that if every element of R is right prime, then R is a division ring, that is, the nonzero elements of R form a group under multiplication. (Note: R is not assumed to be commutative nor is it assumed that R has a multiplicative identity.)

Quickies

Answers to the Quickies are on page 381.

Q985. *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania.*

Let x be a real number. Evaluate the sum

$$\sum_{n=1}^{\infty} n^2 \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Q986. *Proposed by Peter Ross, Santa Clara University, Santa Clara, CA.*

Prove that in a given ellipse, there exist infinitely many inscribed triangles of maximal area.

Solutions

Growth of $\gamma_n - \gamma$ **December 2007**

1781. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Let γ be Euler's constant and for positive integer n define

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n \qquad \text{and} \qquad \alpha_n = 2n(\gamma_n - \gamma).$$

Prove that the sequence $\{\alpha_n\}$ is monotonically increasing and bounded above. In addition, determine $\lim_{n\to\infty} \alpha_n$.

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas G.C., Spain.

For $n \geq 1$, define the sequence $\{\beta_n\}$ by $\beta_n = \gamma_n - \gamma - \dfrac{1}{2n}$. Then, $2n\beta_n = \alpha_n - 1$. For $n \geq 1$, we have

$$\begin{aligned}\beta_{n+1} - \beta_n &= \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) + \frac{1}{2n(n+1)}\\ &= \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) - \frac{1}{2n(n+1)} = f(n),\end{aligned}$$

where

$$f(x)=\frac{1}{x}-\log\left(1+\frac{1}{x}\right)-\frac{1}{2x(x+1)},\quad \text{for } x>0.$$

Considering the derivative of f we find

$$\begin{aligned} f'(x) &= -\frac{1}{x^2}+\frac{1}{x(x+1)}+\frac{2x+1}{2x^2(x+1)^2} \\ &= \frac{-1}{2x^2(x+1)^2}. \end{aligned}$$

Since $f'(x)<0$, it follows that $f(x)$ is decreasing for $x>0$, and consequently, $f(n+1)\le f(n)$ for $n\ge 1$. Because $f(x)\to 0$ as $x\to\infty$, we have $f(n)\ge 0$ for $n\ge 1$. Therefore, $\beta_{n+1}\ge\beta_n$ for $n\ge 1$. Thus β_n is a nondecreasing sequence and it follows that α_n is also nondecreasing.

Next note that

$$f'(x)\sim -\frac{1}{2x^4},\quad x\to\infty.$$

Hence, by integration,

$$f(x)\sim\frac{1}{6x^3},\quad x\to\infty,$$

so $\beta_{n+1}-\beta_n\sim\frac{1}{6n^3}$. Because $\beta_n\to 0$ when $n\to\infty$, it follows by summation that

$$-\beta_n\sim\frac{1}{12n^2}.$$

Therefore, $n\beta_n\to 0$ and consequently, $\alpha_n\to 1$. In addition, we have shown that $n(\alpha_n-1)=2n^2\beta_n\to-\frac{1}{6}$.

Note. Some readers pointed out that the estimate

$$\gamma_n-\gamma=\frac{1}{2n}-\frac{1}{12n^2}+\frac{\epsilon_n}{120n^4},$$

where $0<\epsilon_n<1$, appears on page 264 of the second edition of *Concrete Mathematics*, by Ronald Graham, Donald Knuth, and Oren Patashnik.

Also solved by Michael S. Becker, Khristo Boyadzhiev, John Christopher, Thomas Dence, G.R.A.20 Problem Solving Group (Italy), Kee-Wai Lau (China), Edward Omey (Belgium), Paolo Perfetti (Italy), Jenry Ricardo, Edward Schmeichel, Albert Stadler (Switzerland), David Stone and John Hawkins, Marian Tetiva (Romania), Michael Vowe (Switzerland), Michael Woltermann, and the proposer. There was one incorrect submission.

Determining a length **December 2007**

1782. *Proposed by Stephen J. Herschkorn, Highland Park, NJ.*

Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are perpendicular, D lies on $\overline{BC}$, and E and F lie on $\overline{AC}$. In addition, $\overleftrightarrow{AD}$ and $\overleftrightarrow{DF}$ are perpendicular, $AB=AD=1$, and $AE=DE=x$. Find CF.

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

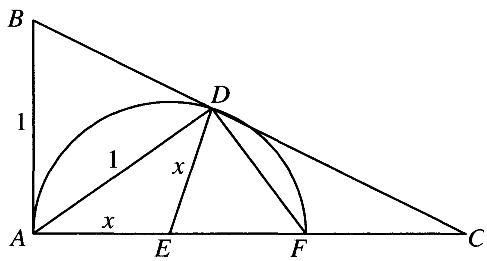
Because triangles ABC and AFD are right and triangle BAD is isosceles, we have

$$\angle FDC = \frac{\pi}{2} - \angle ADB = \frac{\pi}{2} - \angle ABD = \angle FCD.$$

Thus triangle DFC is also isosceles with $CF = DF$. Because $\angle ADF$ is right, the circle of center E and radius x must pass through F. Hence $EF = x$.

Now from right triangle ADF we have

$$CF = DF = 2x\sin(\angle DAE) = 2x\sqrt{1-\cos^2(\angle DAE)} = \sqrt{4x^2-1}.$$



Also solved by Alma College Problem Solving Group, Armstrong Problem Solvers, Herb Bailey, Fidel Barrera-Cruz, Michel Bataille (France), Jany C. Binz (Switzerland), Cal Poly Pomona Problem Solving Group, Robert Calcaterra, Minh Can, Michael J. Caufield, John Christopher, Chip Curtis, Ragnar Dyovik (Norway), Phil Embree, Fejénthaláltuka Szeged Problem Solving Group (Hungary), Marty Getz and Dixon Jones, Michelle Ghrist, Jeff Groah, Bayron Gatiérrez, G.R.A.20 Problem Solving Group (Italy), Brian Hogan, Matthew Hudelson, J&P Group Math Factor, Victor Y. Kutsenok, Math 130 Students at Mary's College of California, Peter Nüesch (Switzerland), Samih Obaid, J. Oelschlager, Samuel Otten, Ángel Plaza (Spain), Kevin Roper, Edward Schmeichel, Seton Hall University Problem Solving Group, Nicholas C. Singer, Skidmore College Problem Group, Ryan Spitler, Albert Stadler (Switzerland), Gail T. Stafford, David Stone and John Hawkins, Awa Traore, Michael Vowe (Switzerland), Stuart V. Witt, Michael Woltermann, Bill Yankosky, Ken Yanosko, Hongbiao Zeng, Chris Zin, and the proposer. There was one solution with no name and five incorrect submissions.

An inequality of reciprocals — December 2007

1783. *Proposed by Ovidiu Bagasar, Babes Bolyai University, Cluj Napoca, Romania.*

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Let $S = x_1^n + x_2^n + \cdots + x_n^n$ and $P = x_1x_2\cdots x_n$. Prove that

$$\sum_{k=1}^{n} \frac{1}{S - a_k^n + P} \le \frac{1}{P}.$$

Solution by Harris Kwong, SUNY at Fredonia, Fredonia, NY.

By the AM-GM inequality

$$\frac{S - x^k + P}{n} = \frac{1}{n}\Big(P + \sum_{\substack{i=1\\ i\neq k}} x_i^n\Big) \ge \Big(P\prod_{\substack{i=1\\ i\neq k}} x_i^n\Big)^{1/n}$$

$$= \left(\left(\frac{P}{x_k}\right)^n \cdot P\right)^{1/n} = \frac{P\sqrt[n]{P}}{x_k},$$

and hence

$$\frac{1}{S - x_k^n + P} \le \frac{1}{n}\frac{x_k}{P\sqrt[n]{P}}.$$

By another application of the AM-GM inequality,

$$\sum_{k=1}^{n} \frac{1}{S - x_k^n + P} \le \frac{1}{P\sqrt[n]{P}} \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \le \frac{1}{P\sqrt[n]{P}} \cdot \sqrt[n]{P} = \frac{1}{P}.$$

We note that equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by Michel Bataille (France), Minh Can, Northwestern University Math Problem Solving Group, Paolo Perfetti (Italy), Henry Ricardo, Albert Stadler (Switzerland), Marian Tetiva (Romania), Bob Tomper, and the proposer.

Integral to series **December 2007**

1784. *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI.*

Let $\alpha > 0$ and let p be a positive integer. Prove that

$$\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(\alpha+p)(2\alpha+p)\cdots(n\alpha+p)} = e\int_0^1 x^{p-1+\alpha} e^{-x^\alpha}\, dx.$$

Solution by Michel Bataille, Rouen, France.
Using the substitution $x = (1-t)^{1/\alpha}$ in the integral, we obtain

$$e\int_0^1 x^{p-1+\alpha} e^{-x^\alpha}\, dx = \frac{1}{\alpha}\int_0^1 e^t (1-t)^{p/\alpha}\, dt = \frac{1}{\alpha}\int_0^1 \left(\sum_{n=0}^{\infty} (1-t)^{p/\alpha} \cdot \frac{t^n}{n!}\right) dt \qquad (*)$$

Let β denote the beta function, defined by $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ for $x, y > 0$. It is well-known that for any nonnegative integer m, we have

$$\beta(m+1, y) = \frac{m!}{y(y+1)\cdot \cdots \cdot (y+m)}.$$

Because the series in (*) converges uniformly on [0, 1], the sum and the integral can be interchanged and it follows that

$$\begin{aligned} e\int_0^1 x^{p-1+\alpha} e^{-x^\alpha}\, dx &= \frac{1}{\alpha}\sum_{n=0}^{\infty} \int_0^1 (1-t)^{p/\alpha} \cdot \frac{t^n}{n!}\, dt = \frac{1}{\alpha}\sum_{n=0}^{\infty} \frac{1}{n!}\beta\left(n+1, \frac{p}{\alpha}+1\right) \\ &= \frac{1}{\alpha}\sum_{n=0}^{\infty} \frac{1}{(\frac{p}{\alpha}+1)\cdot \cdots \cdot (\frac{p}{\alpha}+1+n)} \\ &= \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(\alpha+p)(2\alpha+p)\cdots(n\alpha+p)}. \end{aligned}$$

Also solved by Armstrong Problem Solvers, Paul Bracken, Brian Bradie, Chip Curtis, Costas Efthimiou, João Guerreiro (Portugal), Eugene A. Herman, Kim McInturff, J. Oelschlager, Paolo Perfetti (Italy), Robert W. Pratt, Kevin Roper, Nicholas C. Singer, Dmitri V. Skjorshammer, Albert Stadler (Switzerland), Marian Tetiva (Romania), Bob Tomper, Michael Vowe (Switzerland), and the proposer.

Summing floor powers **December 2007**

1785. *Proposed by Mihaly Bencze, Brasou, Romania.*

Let k be a positive integer, let x a real number, and let $\{x\}$ denote the fractional part of x. Prove that

a. $\displaystyle\sum_{j=1}^{n}\left\lfloor x+\frac{j-1}{n}\right\rfloor^{k}=n\lfloor x\rfloor^{k}+\left((\lfloor x\rfloor+1)^{k}-\lfloor x\rfloor^{k}\right)\lfloor n\{x\}\rfloor.$

b. $\displaystyle\sum_{j=1}^{n}\left\lfloor x+\frac{2j-1}{2n}\right\rfloor^{k}=n\lfloor x\rfloor^{k}+\left((\lfloor x\rfloor+1)^{k}-\lfloor x\rfloor^{k}\right)\left\lfloor n\{x\}+\frac{1}{2}\right\rfloor.$

Solution by Marty Getz and Dixon Jones, University of Alaska Fairbanks, Fairbanks, AK.

We prove that for real $r\geq 1$,

$$\sum_{j=1}^{n}\left\lfloor x+\frac{rj-1}{rn}\right\rfloor^{k}=n\lfloor x\rfloor^{k}+\left((\lfloor x\rfloor+1)^{k}-\lfloor x\rfloor^{k}\right)\left\lfloor n\{x\}+\frac{r-1}{r}\right\rfloor.$$

For $r=1$, the sum calls for the floors of the n numbers

$$x,x+\frac{1}{n},x+\frac{2}{n},\dots,x+\frac{n-1}{n}.$$

We will count the number m of these values that occur in the interval $[\lfloor x\rfloor+1,x+1]$. This will give

$$\begin{aligned}\sum_{j=1}^{n}\left\lfloor x+\frac{j-1}{n}\right\rfloor^{k}&=(n-m)\lfloor x\rfloor^{k}+m\,(\lfloor x\rfloor+1)^{k}\\&=n\lfloor x\rfloor^{k}+m\left((\lfloor x\rfloor+1)^{k}-\lfloor x\rfloor^{k}\right).\end{aligned}$$

To calculate m, we multiply each of the numbers in question by n then seek the m numbers of the list $nx, nx+1, nx+2,\dots,nx+(n-1)$ that are in the interval $[n\lfloor x\rfloor+n, nx+n]$. The length of this interval is $n\{x\}$. By counting leftward from $nx+n$, we see that $m=\lfloor n\{x\}\rfloor$.

For $r\geq 1$, the number of points from the list

$$x+\frac{r-1}{rn},x+\frac{2r-1}{rn},x+\frac{3r-1}{rn},\dots,x+\frac{nr-1}{rn}$$

that are in the interval $[\lfloor x\rfloor+1,x+1]$ may be counted by a similar method: it is the number m of points from the list

$$nx+\frac{r-1}{r},nx+\frac{r-1}{r}+1,nx+\frac{r-1}{r}+2,\dots,nx+\frac{r-1}{r}+(n-1)$$

in the interval $[n\lfloor x\rfloor+n, nx+n]$. Noting that the list is simply a rightward translation through distance $\frac{r-1}{r}$ of the list for the case $r=1$, we find

$$m=\left\lfloor n\{x\}+\frac{r-1}{r}\right\rfloor,$$

and the result follows.

Also solved by Michel Bataille (France), Jany C. Binz (Switzerland), Brian Bradie, John Christopher, Chip Curtis, Dmitry Fleischman, Hyun Soo Park (Korea), Robert W. Pratt, Albert Stadler (Switzerland), Stuart V. Witt, and the proposer.

Answers

Solutions to the Quickies from page 376.

A985. Let $S(x)$ be the sum of the series. Then, by differentiation,

$$S'(x) = \sum_{n=1}^{\infty} n^2 \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^{n-1}}{(n-1)!} \right) = S(x) + \sum_{n=1}^{\infty} n^2 \frac{x^n}{n!}$$

$$= S(x) + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}(n-1+1) = S(x) + \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}.$$

It follows that $S'(x) = S(x) + x^2 e^x + x e^x$, and hence

$$S(x) = \left(\frac{x^3}{3} + \frac{x^2}{2} \right) e^x + C e^x,$$

where C is a constant of integration. Because $S(0) = 0$, we have $C = 0$ and

$$S(x) = \left(\frac{x^3}{3} + \frac{x^2}{2} \right) e^x.$$

A986. We first solve the problem for a circle. For a circle, the inscribed triangles of maximal area are equilateral triangles. To prove this, consider a chord C of the circle. The inscribed triangle of maximal area with C as a side is isosceles, because such a triangle maximizes the altitude for base C. In addition, given a point p on the circle there is a unique equilateral triangle with p as a vertex.

The linear transformation $T(x, y) = (ax, by)$, $a, b > 0$ maps the unit circle to an ellipse with semi-axes a and b and has constant, nonzero Jacobian ab. Thus, for any point P on the ellipse, the image under T of the inscribed equilateral triangle with vertex $p = T^{-1}(P)$ will be an inscribed triangle of maximal area. In particular, there are infinitely many such triangles and, if $a \neq b$, they are from infinitely many different congruence classes.

Because invertible linear transformations take parallel lines to parallel lines, a similar argument shows that in a given ellipse there are infinitely many inscribed parallelograms of maximal area, and given any point p on the ellipse, there is a unique such parallelogram with vertex p.

Editor's Note. In the December 2007 issue, Albert Stadler of Switzerland should have been listed among the solvers of problem 1756. In the October 2008 issue, Michel Bataille of France should have been listed among the solvers of problems 1776, 1777, 1778, and 1779.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Defense Sciences Office, Broad Agency Announcement: DARPA Mathematical Challenges, DARPA-BAA 08-65 September 26, 2008, `https://www.fbo.gov/download/9bc/9bce380aafb19f9ad3bda188bfc1ab20/DARPA-BAA-08-65.doc`.

The Defense Advanced Research Projects Agency (DARPA) is soliciting research proposals "with the goal of dramatically revolutionizing mathematics and thereby strengthening scientific and technological capabilities." Parallel to Hilbert in 1900, it identifies 23 "mathematical challenges," each described in only a pair of lines or so. Yes, the Riemann Hypothesis is there (#19), but more representative of the generality is #15: "The Geometry of Genomic Space: What notion of distance is needed to incorporate biological utility?" The deadline for proposal abstracts is 25 July 2009, but the level(s) of funding and project duration are unspecified. The second criterion after "scientific and technical merit" is "long term value to defense," which certainly must be taken in a very wide sense to embrace many of the challenges listed.

Hansell, Saul, How Wall Street lied to its computers, `http://bits.blogs.nytimes.com/2008/09/18/how-wall-streets-quants-lied-to-their-computers/`.

Chu-Carroll, Mark C., Bad probability and economic disaster; or how ignoring Bayes theorem caused the mess, `http://scienceblogs.com/goodmath/2008/09/bad_probability_and_economic_d.php`.

Rickards, James G., A mountain, overlooked, *Washington Post* (2 October 2008) A23, `http://www.washingtonpost.com/wp-dyn/content/article/2008/10/01/AR2008100101149.html`.

Ellenberg, Jordan, We're down $700 billion. Let's go double or nothing!, Slate (2 October 2008) `http://www.slate.com/id/2201428/`.

Did mathematics cause the economic crisis? Mathematics can guide any kind of optimization, and in recent years Wall Street has hired (with handsome salaries) some of our best students to do computer modeling and "financial engineering." Hansell states that the models underestimated risk but only because bankers fed them overoptimistic assumptions. Chu-Carroll says that "they cheated in the math," meaning that the probabilities of loss, default, and disaster were calculated on the false assumption that loan failures are stochastically independent. Rickards echoes that but also suggests mathematical chaos arising out of complexity. He asserts that the predominant "value at risk" (VaR) model, which assesses overall risk by aggregating over numerous tiny risks, is completely "the wrong paradigm" and that there is no hope that Wall Street and its regulators can avert catastrophes until they abandon it. Mathematician Ellenberg likens financial derivatives to martingales and notes that "unless some real pain for the martingalers is built in, we'd better be ready for a return to maverick finance down the road." A respondent to Chu-Carroll's blog suggests that bankers "ignore[d] the math and picked the option they wanted without concern for risk or future events unrelated to them." Well, our schools and culture teach much less about pursuing the common good (that is left to the churches) than "the American dream" of unlimited economic opportunity. For some, that dream becomes avarice, the "immoderate desire for wealth." As a tool, mathematics is not in a position to object to the purposes to which it is put. Objection to greed—and the perversion of mathematics to that end—needs to come from the churches, the schools, and the culture, including mathematicians. We need to provide guidance to students about worthy enterprises for their skills.

Guzdial, Mark, The big ideas of computer science, Part 1 and Part 2 (9 September 2008), `http://www.amazon.com/gp/blog/post/PLNK1KCVVK85JUI3H`, `/PLNK2UHDL465MED27`;

Programming is central to computer science (25 September 2008), `/PLNK3916UJMECC77L`;

Programming is central to computer science, and we can change programming (2 October 2008), `/PLNK1VTB7QIX91BSE`.

Computer science unplugged: Teach computer science without a computer!, `http://csunplugged.org/`.

What is important in learning computer science? What are the most significant concepts and activities? Alan Perlis's list in 1961 will remind mathematicians of important ingredients in a mathematical education, too: parametrization, iteration, recursion, definitions, attention to eventualities regardless of likelihood, representation, language, simulation, and proof. Additions by author Guzdial are more equipment-oriented: sequential instructions, coding time to exit, memory, layering, methods to invoke instructions, protocols and standards, and flawed humans. An evergreen question in computer science education—whether programming is central, whether one can learn much about computer science without programming—is the subject of his latter two blogs. "'Breadth-first' computer science (teaching about all of CS, without programming) has virtually disappeared in computer science programs. Students found it boring, irrelevant, and lacking the feedback.... We need to distinguish programming from the task of being a programmer. We do need people to be programmers, but that's not an attractive job for many.... The difference between programming and being a programmer is the same as between writing and being a novelist. Everyone should know how to write."

Gowers, Timothy, June Barrow-Green, and Imre Leader (eds.), *The Princeton Companion to Mathematics*, Princeton University Press, 2008; xx+1034 pp, $99. ISBN 978-0-691-11880-2.

I am a priori skeptical of mathematical dictionaries, compendia, encyclopediae, handbooks, "companions," and the like. I have several but refer to them rarely and usually without being satisfied. They often seem intended for libraries, which feel obliged to have *some* reference work on mathematics. Moreover, there are numerous Websites, including Wolfram MathWorld (`http://mathworld.wolfram.com`). The preface of this book addresses that competition by stressing the book's long essays and carefully ordered sequence. As a "companion," the book focuses on "modern, pure mathematics"; does not attempt to be encyclopedic; is organized thematically rather than alphabetically; and has a relatively low ratio of symbols to prose. Its 200 contributors include dozens of names that you would recognize. The level of difficulty is not uniform, but "the editors have tried very hard not to allow any material into the book that they do not themselves understand, which has turned out to be a very serious constraint." Apart from an introduction, the book has sections on the origins of modern mathematics, mathematical concepts, branches of mathematics (one-third of the book), theorems and problems, mathematicians, the influence of mathematics, and final perspectives. You can't take in all of such a book at once; but as I use it, I am beginning to appreciate it.

Peterson, Ivars, Improved pancake sorting (9 October 2008) `http://www.maa.org/mathtourist/mathtourist_10_0_08.html`.

Malkevitch, Joseph, Pancakes, graphs, and the genome of plants, *The UMAP Journal of Undergraduate Mathematics and Its Applications* 23 (4) (2002) 373–382.

Hayes, Brian, Sorting out the genome, *American Scientist* 95 (September-October 2007) 286–391.

Random reversals of blocks of genes can produce variations in organisms, and the number of reversals is a clue to how distantly related organisms are. "Gene flipping" is related to "pancake flipping," putting a stack of pancakes in order of size by a sequence of moves of inserting a spatula, lifting off the pancakes above, and replacing them in reverse order. Peterson gives news of an improved upper bound, $18n/11$, for the number of flips to order the stack, proved by students at the University of Texas at Dallas. Malkevitch's clear expository essay describes the pancake problem and its history, suggests undergraduate research projects, and mentions applications in computer design of the "pancake topology" of processors. Hayes's delightful article sticks closely to the biological applications of variations on the pancake problem. This is the only subject on which Bill Gates, as a student, published a scientific paper.

NEWS AND LETTERS

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Aboufadel, Edward F., *Grand Valley State University, Allendale, MI*

Ash, J. Marshall, *De Paul University, Chicago, IL*

Barnes, Julia, *Western Carolina University, Cullowhee, NC*

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Johnston, Elgin, *Iowa State University, Ames, IA*

Kallaher, Michael, *Pullman, WA*

Katz, Victor J., *Silver Spring, MD*

Kendig, Keith M., *Cleveland State University, Cleveland, OH*

Kilgour, Mark, *Wilfrid Laurier University, Waterloo, ON, Canada*

Kung, Sidney, *Cupertino, CA*

Luca, Florian, *University Nacional Autonoma de Mexico Mathematical Institute, Morelia, Mexico*

McCarthy, John E., *Washington University, St. Louis, MO*

McCooey, Michael P., *Franklin and Marshall College, Lancaster, PA*

McLaughlin, Jim, *West Chester University of Pennsylvania, West Chester, PA*

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Moses, Peter, *Moparmatic, CO, Astwood Bank, Worcestershire, England*

Narayan, Darren, *Rochester Institute of Technology, Rochester, NY*

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Neugenbauer, Christoph J., *Purdue University, West Lafayette, IN*

Nelsen, Roger, *Lewis and Clark College, Portland, OR*

Oakley, Patricia, *Goshen College, Goshen, IN*

Passell, Nicholas, *University of Wisconsin-Eau Claire, Eau Claire, WI*

Pedersen, Jean J., *Santa Clara University, Santa Clara, CA*

Penn, Howard, *United States Naval Academy, Annapolis, MD*

Rose, David, *Florida Southern College, Lakeland, FL*

Rosenstein, George, *Lancaster, PA*

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Roth, Kimberly, *Juniata College, Huntingdon, PA*

Schilling, Mark, *California State University Northridge, Northridge, CA*

Scott, David R., *University of Puget Sound, Tacoma, WA*

Shell-Gellash, Amy, *Pacific Lutheran University, Tacoma, WA*

Sizer, Walter, *Minnesota State University Moorhead, Moorhead, MN*

Soifer, Alexander, *University of Colorado at Colorado Springs, Colorado Springs, CO*

Sondow, Jonathan, *New York, NY*

Spivey, Michael, *University of Puget Sound, Tacoma, WA*

Stahl, Saul, *University of Kansas, Lawrence, KS*

Stockmeyer, Paul, *College of William and Mary, Williamsburg, VA*

Wagon, Stanley, *Macalester College, St. Paul, MN*

Walsh, Dennis, *Middle Tennessee State University, Murfreesboro, TN*

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Wardlaw, William, *United States Naval Academy, Annapolis, MD*

Watkins, John J., *Colorado College, Colorado Springs, CO*

White, Arthur T., *Western Michigan University, Kalamazoo, MI*

Williams, Gordon, *Moravian College, Bethlehem, PA*

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AUTHORS

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THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, NW
Washington, DC 20036